Willment, Diana, 1985.
Complex numbers from 1600 to 1840.
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COMPLEX NUMBERS FROM 1600 TO 1840

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This thesis is submitted to the CNAA in partial fulfilment of the requirements for the degree of Master of Philosophy

Sponsoring establishment : Middlesex Polytechnic

January 1985

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ABSTRACT

This thesis uses primary and secondary sources to study advances in complex number theory during the 17th and 18th Centuries. Some space is also given to the early 19th Century. Six questions concerning their rules of operation, usage, symbolism, nature, representation and attitudes to them are posed in the Introduction. The main part of the thesis quotes from the works of Descartes, Newton, Wallis, Saunderson, Maclaurin, d'Alembert, Euler, Waring, Frend, Hutton, Arbogast, de Missery, Argand, Cauchy, Hamilton, de Morgan, Sylvester and others, mainly in chronological order, with comment and discussion. More attention has been given to algebraists, the originators of most advances in complex numbers, than to writers in trigonometry, calculus and analysis, who tended to be users of them. The last chapter summarises the most important points and considers the extent to which the six questions have been resolved. The most important developments during the period are identified as follows :

- (i) the advance in status of complex numbers from 'useless' to 'useful'
- (ii) their interpretation by Wallis, Argand and Gauss in arithmetic, geometric and algebraic ways
- (iii) the discovery that they are essential for understanding polynomials and logarithmic, exponential and trigonometric functions
- (iv) the extension of trigonometry, calculus and analysis into the complex number field
- (v) the discovery that complex numbers are closed under exponentiation, and so under all algebraic operations
- (vi) partial reform of nomenclature and symbolism
- (vii) the eventual extension of complex number theory to n dimensions

In spite of the advances listed above, it is noted that there was a continued lack of confidence in complex numbers and avoidance of them by some mathematicians, particularly in England.

Introduction

The history of the number system does not neatly follow the modern set diagram for the complex number field, in order of discovery, acceptance or definition. Natural numbers were discovered first, probably soon after language started, but defining axioms for them appeared last (Peano's axioms, 1889). Positive fractions and irrationals were next in common use, and the difference between them first noticed by the Pythagoreans about 600 B.C. Fractions could not be defined until naturals were defined but a working definition was possible which described them as the ratio of two naturals. Irrationals were described by the Greeks in terms of what they were not, that is they were not commensurable with the naturals. In this is the germ of the 19th Century idea that all reals except naturals and those dependent on them for their definition (integers and rationals), are irrationals. The nature of transcendentals as different from algebraic irrationals was discovered in the 18th Century, but an acceptable definition for irrationals was not given until the 1870's, by Dedekind and others. Negative and complex numbers were accepted reluctantly from Renaissance times onwards. Negative integers can be defined by extending the naturals to zero and beyond, and modifying Peano's axioms. Imaginary numbers can be defined once the reals are complete, and complex numbers when both real and imaginary numbers are defined. A clear picture of the way in which reals are distributed on the number line depends also on insight into transfinite numbers. The Greeks left a legacy of evasion of the infinite, both in number and magnitude, and some mathematicians (Gauss, Cauchy) denied that an infinite set could exist, while others ignored the whole difficult topic. Cantor was able to give a definition of an infinite class during the 1870's and was able to use his ideas to describe the distribution of integers, rationals and irrationals on the real number line. A similar set of imaginaries with similar properties could be represented on a perpendicular axis in the manner discovered by Wessel and Argand, to define the complex plane. Except for the irrationals, all the number categories depended on definition of the naturals. It can be said that with Peano's axioms of 1889, the number system was fully described, and could be represented on a complex plane with defined properties. The picture of the complex number field as nested sets could be given at about the same time as the definitions. However number subsets were being widely used long before these definitions and descriptions were given, which shows not only a great pioneering spirit among mathematicians, but great confidence in the structure of mathematics and its procedures.

This work covers two hundred years of development in complex number theory and traces an important advance in their standing. At the end of this period the subsets of the complex number field were known of and the way was clear for 19th Century mathematicians to clarify and simplify the situation by providing defining axioms and a general description of the number system.

This investigation was undertaken with the objective of tracing developments in the use, theory and status of complex numbers during the 17th and 18th Centuries.

This period in the history of complex numbers is an important one as it saw great advances in their status and place in mathematics. The rules for addition, subtraction, multiplication and division of complex numbers and the behaviour of conjugates were known before the end of the 16th Century, but they were only being used in the solution of equations. By the beginning of the 19th Century they were being integrated into calculus, trigonometry and the theory of logarithmic and exponential functions, and were known to be closed under exponentiation. It is remarkable that, despite the great progress made during this period, there was a widespread lack of confidence, not only in complex numbers but in negatives also.

The aspects of complex numbers that will be considered can be summarised as follows :

(i) the emergence of rules of operation

(ii) ways in which they were used, both in the solution of problems and as part of the fabric of mathematics

(iii) $\sqrt{-1}$ as a symbol and interactions with other symbols such as D, the differential operator

(iv) the nature of $\sqrt{-1}$, both in a metaphysical sense (what kind of entity is it, does it actually exist ?), and in a mathematical sense (is it algebraic, geometric, arithmetic ?)

(v) the search for a physical model or geometrical representation

(vi) their perceived status and attitudes to these numbers

After Bombelli's Algebra (1572), little was added to the rules for the arithmetic of complex numbers until Euler gave a value for $(\sqrt{-1})^{\sqrt{-1}}$ in 1746, and the work of d'Alembert, Euler, Lagrange and others in proving that $(a + \sqrt{-1}b)^{g + \sqrt{-1}h}$ is a complex number of the form $p + \sqrt{-1}q$, so the complex number field is closed under algebraic operations. As discoveries were made about more advanced concepts such as the logarithms of negative numbers and complex series, the rules of behaviour of complex numbers evolved on the basis of consistency with the reals. Complex numbers were being used with much success to solve problems, particularly theoretical mathematical ones, and this emphasised the need to clarify the status of complex numbers. Euler gave thought to the nature of complex numbers, and although it did not prevent him from making major breakthroughs on such problems as the logarithms of negative numbers, he felt his own lack of insight. In the search for models and representations the most successful mathematician during this period was Wallis, who devised both a primitive form of the Argand diagram and a definition of $\sqrt{-1}$ in terms of mean proportionals. The attitudes of mathematicians can be found not only in what they wrote, but in what they did not write. It is possible to divide mathematicians into those who gave complex numbers some kind of coverage, and those who sometimes or always ignored them. In the case of Charles Hutton, it has been possible to infer that his omission of the topic from an otherwise comprehensive text-book was due to his encountering some misleading information in Euler's Algebra. The lack of a visual representation for $\sqrt{-1}$ had a profound influence on attitudes to it, and complex numbers were not widely accepted until after the invention of the Argand diagram.

It is clear that acceptance of complex numbers percolated only slowly through the mathematical world. A mass of comment has been collected, expressing bewilderment and exasperation with entities that lent themselves to useful mathematical development, but whose nature was obscure. The formalist view had not been described at this time, and mathematicians found the situation so intolerable that some tried to ignore complex numbers, and others when giving proofs involving them, gave alternative proofs often much longer.

The period has been divided into sections chronologically. The first chapter summarises the situation with regard to irrational, negative and complex numbers at the beginning of the 17th Century. Negatives have been included because understanding these is essential to understanding complex numbers. Irrationals have been included not only because they are a part of the number system, but because attitudes to them passed through similar stages to attitudes to negative and complex numbers. It may be said that the difficulties with complex numbers constituted the third crisis of confidence to occur in the development of the number system. Fractions (rationals) have not been dealt with as acceptance of these has not caused similar difficulties to mathematicians.

Chapter II covers the period from the beginning of the 17th Century to the work of John Wallis. Primary sources used include work by Descartes, Newton and Wallis.

Chapter III continues from the time of Wallis to that of Leonhard Euler. Primary sources include work by Saunderson, d'Alembert, Maclaurin and Euler.

Chapter IV covers the period from Euler to the beginning of the 19th Century, ending with the Wessel/Argand diagram. The main primary sources are works by Waring, Frend, Hutton, Lagrange, Laplace, Arbogast, de Missery and Argand.

Chapter V describes the position in the early 19th Century. It refers to work by Cauchy, Hamilton and de Morgan.

The summary includes a consideration of the extent to which the original points have been answered. Some suggestions for further research have been put forward in Appendix II.

I have sometimes referred to $\sqrt{-1}$ as 'i' and used the term 'complex number', although these terms were not in widespread use before the 19th Century. I have also used such terms as polynomial, field, operation etc in their modern sense.

Sources used for this investigation include text-books, histories, biographies, encyclopaedias, dictionaries, periodicals and correspondences. Where works were not originally published in English, English translations have been used when available, and treated as primary sources. A question which must be considered concerns the extent to which the primary sources can be expected to shed light on the topic under investigation. Text-books covering complex numbers generally contain rules of procedure, making it fairly easy to deal with the first point, although this area has already been well covered. However, where text-books are the only source of information, the author's opinions about complex numbers are not often clear, such a book would not reveal any unusual views held by the author. Where the topic has been omitted, possible reasons may have to be guessed at. Encyclopaedias and dictionaries have proved useful in revealing attitudes to complex numbers.

The search for attitudes to complex numbers has focussed attention on algebraic works. It was found that users of complex numbers in calculus, analysis, trigonometry etc remained just that; tending to accept them in a formalist way and incorporating them into many branches of mathematics as entities conforming to known rules. It was the algebraists who were concerned with their nature and status and who provided the most interesting insights into the difficulties as they saw them. For this reason, in this investigation, more attention has been paid to writings on algebra than to those on analysis etc.

I should like to acknowledge my indebtedness to Mathematical Thought from Ancient to Modern Times (1972) by Morris Kline, which was used as a starting point and principal secondary source. Where sources are not cited, this book has been relied upon for information.

I am indebted to the following libraries for access to books :

Science Museum
Senate House, University of London
Royal Society
West London Institute of Higher Education
Warburg Institute, University of London

I should also like to thank the following for their help and advice :

Dr I. Grattan-Guinness
Mr S.A. Jayawardene
Prof C.W. Kilmister

Chapter I

Negative, irrational and complex numbers before the early 17th Century

The four basic rules for the arithmetic of complex numbers and the additive and multiplicative properties of conjugates were available by 1600. The Algebra of Raphael Bombelli (published 1572, MS earlier) gave these rules correctly.

The most serious difficulty hindering understanding of complex numbers at this time, and for long after, was that negative numbers were not yet accepted nor fully understood. Cardano referred to them as 'ficticious', Chuquet and Stifel as 'absurd' and even Descartes as 'false'. Vieta totally ignored negatives in his algebra book, the Arithmetica Speciosa of 1590. Girard gave negatives equal treatment with positives and Bombelli gave definitions of them although he was more adventurous with imaginaries than with negatives. Harriot experimented with different rules for '- times +', and '- times -', and explored an unorthodox algebra arising from taking as axiomatic that '- times - gives -', with the help of a special symbol to denote the product '- times +' (1). The difficulty arose from the fact that the first printed version of Bombelli's Algebra had some vital and misleading inaccuracies, although the MS version was correct. These errors were rectified later, but meanwhile the statement '- times - gives -' caused confusion to both Harriot and Cardano. However Harriot did not accept negatives as roots although he used negative roots in equations, and sometimes even a negative quantity alone on one side of an equation. Stevin, by using negative coefficients in equations, gave a single method for the solution of quadratics using algebraic methods to prove his solution correct, but gave only real roots. Descartes changed his attitude to negatives when he discovered that negative roots may be increased by any desired amount and become positive, by simple real manipulations of the unknown.

(1) Tanner, "The Ordered Regiment of the Minus Sign", Annals of Sc., 37(1980),127-58,(p.134)

However this procedure was not known in the 16th Century, and if it had been, might only have increased suspicion of complex roots (as it did for Descartes), as these roots are not susceptible to same treatment.

In the Arithmetica Integra of 1544, Vieta accepts irrationals only reluctantly and on the grounds that the results obtained from using them are valid. This argument has frequently been used by mathematicians to justify acceptance of number categories, particularly complex numbers. Vieta takes the view that irrationals, like infinity, are not true numbers as they are not exactly expressible as decimals. Later, Pascal, Barrow and Newton were to accept irrationals as geometric magnitudes, justifiable by the Eudoxan theory of magnitudes.

Attitudes to complex numbers in the 16th Century were even less confident. Cardano refers to 'mental tortures' and calls a complex solution 'as refined as it is useless'. Bombelli, although giving the four rules correctly and appearing to manipulate complex numbers confidently (especially conjugates), nevertheless refers to them as 'useless' and 'sophistic'. This was not an unusual view, their usefulness was only recognised later. Bombelli was the first to show that Cardano's method for the roots of a cubic gave a real root in the irreducible case, where the root is a complicated complex expression. Bombelli used geometrical methods for his proof; Vieta and Girard later used trigonometry. This did not remove the paradox that a complicated expression involving the cube roots of complex numbers should reduce to a real number. It was shown that it did in fact do so, but this still seemed paradoxical. This demonstration was not the powerful argument for generating confidence in negative and complex numbers that it should have been. Bombelli also solved certain quartics, showing courage in manipulating symbols whose meanings may not be easy to interpret. Cardano was able to manipulate complex numbers to some extent, but had little confidence in them. Harriot said that every quadratic 'if possible' has two real roots, 'but in case the Equation be impossible, those two roots are not Real but only Imaginary'[(1)]. Harriot did not give any complex roots, to him 'imaginary' meant 'non-existent'.

(1) Wallis, Algebra , p.132

Girard stated without proof that every polynomial of degree n has n roots, implying acceptance of negative and complex quantities as roots to be counted together with positive real ones and taking repeated roots as separate. Counting these together shows that he thought of them as the same kind of entity, but he would not necessarily have had any very sophisticated ideas about only counting together homogeneous quantities. Magnitudes, areas and volumes had been added in equations since Greek times, at least. Girard's views do not seem to have had much impact on mathematical thinking.

The fears and misunderstandings prevalent in the 16th Century about negative and complex numbers can be attributed only partly to the fact that the second were being forced upon the attention before the first had been assimilated. They must be traced back, at least in part, to the traditional practice of proving real number algebraic results by geometric Euclidean methods. It was not until the 19th Century that a partially successful structure was devised to give algebra a rigour comparable with that attributed to geometry. The Greek legacy was not only a wealth of ideas for rediscovery, but the restriction of an algebra based on geometric magnitudes requiring geometric proofs, rather than on numbers. Because of the difficulty of geometric representation, this severely hampered understanding of negative and complex numbers, and so of full understanding of the number system. A further serious difficulty was the proliferation of algebraic notations. Many of these involved abbreviations of Latin words which had not been well-considered, and the number of variations indicates their unsatisfactoriness. This problem was not resolved until the publication in 1637 by Descartes of La Géométrie. By then printing had been developing for about two hundred years and the lucid notation of Descartes, only partly his own, was taken up and little altered from that time. A further obstacle was a lingering tendency to secrecy about new mathematical discoveries persisting from Medieval times. This practice was dying out by the end of the 16th Century, but the founding of learned societies and journals for free exchange of information did not take place until the second half of the 17th Century.

Finally, the names 'imaginary' and 'impossible' in use at this time inevitably give the impression that the writers thought of them as actually imaginary or impossible, that is non-existent. Unfortunately the names reinforced this impression for readers at the time, in a vicious circle that would have been very difficult to break out of. If the nomenclature had been thought unsuitable then more suitable neutral names would have been devised for them. Once such names become attached to number categories, it is difficult to see how they could have been thought of in any new, constructive or abstract way. In the case of complex numbers, these would have been taken as formal solutions to equations which could have been useful in certain ways, but not as actual answers to problems with any sort of existence. Even the name 'complex' is not a very great improvement, it perpetuates the notion that the number system encompasses some very abstruse ideas. This point is only important if the naturals, say, are thought to have some ideal Platonistic existence somewhere. It is only if this view of numbers is taken that it is important whether complex numbers, say, have that kind of existence. All numbers must participate in the same kind of existence if they are to be the same kind of entity. To many mathematicians the existence or otherwise of numbers depended on whether they were geometrically constructable. This is one of the reasons why the Argand diagram later became important, but this was not to come for another two centuries.

The general picture at the beginning of the 17th Century was one in which irrationals were barely acceptable, negatives were very reluctantly accepted and only because thought useful, but complex numbers, if they had any existence at all, were considered strange and useless.

Chapter II

The early 17th Century to the Algebra (1685) of John Wallis

This period saw understanding of negative and complex numbers progress to a point where John Wallis, in his Treatise of Algebra of 1685, was able to give a diagrammatic representation of a complex number, an explanation of $\sqrt{(-1)}$ in terms of mean proportionals and attempt a concrete interpretation by means of problems which gave rise to complex answers. But this point was not attained easily, nor did these achievements of Wallis have the impact that might have been hoped for.

Among discussions during this period about negative numbers was the paradox of Arnauld which questioned the equality $-1 : +1 = +1 : -1$; since -1 is less than $+1$ how can a smaller quantity to a greater be equal to a greater to a smaller ? No satisfactory answer to this question was produced, -1 cannot be equal to $+1$ and the paradox is not removed by taking -1 to be greater than $+1$. Moreover Wallis, in his Arithmetica Infinitorum (1655) suggests that negatives must be greater than infinity. Here Wallis argued that since $a/0$ with positive a is infinite then when the denominator is negative, a/b with negative b must be greater than infinity. Considering the following sequences:

$$A = \ldots \frac{1}{2} \quad \frac{1}{1} \quad \frac{1}{\cdot 5} \quad \frac{1}{\cdot 25} \quad \frac{1}{0} \quad \frac{1}{-\cdot 25} \quad \frac{1}{-\cdot 5} \ldots$$

and $$B = \ldots \cdot 5 \quad 1 \quad 2 \quad 4 \quad \infty \quad -4 \quad -2 \ldots$$

we have the denominators in sequence A progressively decreasing from positive through zero to negative, and the corresponding values in sequence B progressively increasing from positive through infinity to become negative. This was Wallis's argument that negatives must be greater than infinity. However Wallis seems to have overlooked the fact that the the first premise, that the denominators in sequence A are decreasing, assumes that negative numbers are less than zero.

So, by this argument, the premise that negatives are less than zero leads to the conclusion that they are greater than infinity. Wallis does not mention this serious paradox, he may not have considered it a paradox or possibly may not have noticed it. Wallis simply concludes that negative numbers are greater than infinity rather than that the assumption that they are less than zero leads to the conclusion that they are greater than infinity. This must be a paradox if not a contradiction. Paradoxes of this kind arise from the nature of the discontinuity at 1/0 and the difficulties of reconciling the order and ratio relations, and reinforced the doubts about negative numbers which continued to hinder the acceptance of complex numbers.

Descartes considers quadratic equations with irrational coefficients (see below), and Wallis, in Chapter XLVII of his Algebra [(1)] uses the method of separation of rational and irrational parts when evaluating a supposed root of a polynomial. This parallels the separation of real and imaginary parts by later mathematicians such as Euler. In Chapter LXVI Wallis takes the method of finding the square root of a positive quantity by mean proportionals and extends it to the square roots of negative quantities[(2)]. That is, once again, a method in use for finding an irrational quantity was being used to gain insight into complex quantities (see below).

References to complex numbers during this period refer mainly to their part in the solution of the cubic, the modification of quadratics by real number operations, the numbers of roots in equations and the nature of 'impossible' quantities.

René Descartes 1597-1650

Descartes published La Géométrie in 1637. In it is described the method for modifying the roots of a quadratic by addition or multiplication without evaluating them. He points out the unsatisfactory fact that the manipulations described can make negative roots positive but cannot eliminate complex roots. He uses real manipulations

(1) Wallis , pp.177-79

(2) Wallis , p.266

of the unknown, overlooking the fact that only complex manipulations can eliminate complex roots. Bearing in mind that the complex roots of a quadratic equation with integral coefficients must be conjugates, the procedure is to add or subtract the appropriate imaginary quantity in such a way as to remove it. Since $x^2 - 2ax + a^2 + b^2 = 0$ has complex roots α and β where $\alpha = a + ib$ and $\beta = a - ib$, we must form the equation with roots $\alpha - ib$ and $\beta + ib$. This means that, not only must we know the exact imaginary quantity to add and subtract, but the addition and subtraction must be done so that $\alpha - \beta = 2ib$ and not $-2ib$. Using $\alpha\beta = a^2 + b^2$ and $\alpha + \beta = 2a$, this leads to $x^2 - 2ax + a^2 = 0$ which is the equation required, having two equal real roots. This is equivalent to eliminating from the equation the quantity which prevents it from being a perfect square. Descartes might not have considered this a valid expedient, even with the explanation above. The fact that negative roots can be eliminated from a quadratic equation by real operations on the unknown, but complex ones cannot, had the dual effect (for Descartes and for others) of raising confidence in negatives but reducing confidence in complex numbers.

Descartes first gives the the number of roots of a polynomial as the 'number of dimensions of the unknown quantity' in Book III[1]. He then observes that the degree of an equation can be reduced by division by $x - \alpha$ where α is a known root. His next point is the so-called 'Rule of Signs' for the numbers of positive and negative roots but without any proof or explanation (p. 373) :

> 'We can determine also the number of true and false roots that any equation can have, as follows; An equation can have as many true roots as it contains changes of sign, from + to - or from - to + ; and as many false roots as the number of times two + or two - signs are found in succession.'

He then comes to the method of modifying the roots, using as a first example the polynomial whose roots are 2, 3, 4 and -5. He points out that it is not necessary to know the roots and gives further examples in which these are not known. The rule given for modification of roots would apply equally to real or complex manipulations, although Descartes

(1) Descartes, Geometry (1954), p. 372

is evidently thinking of real ones (p. 373) :

> 'It is also easy to transform an equation so that all the roots that were false shall become true roots, and all those that were true shall become false. This is done by changing the signs of the second, fourth, sixth, and all even terms, leaving unchanged the signs of the first, third, fifth, and other odd terms. Thus, if instead of
> $+x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$ we write $+x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$ we get an equation having one true root, 5, and three false roots, 2, 3, and 4.
>
> If the roots of an equation are unknown and it be desired to increase or diminish each of these roots by some known number, we must substitute for the unknown quantity throughout the equation, another quantity greater or less by the given number.'

Descartes also notes (p. 375) that by a suitable choice of number to add to the unknown, a root may be made equal to zero. He next gives two reasons why he sees these manoeuvres as important. Firstly it is possible to eliminate the second term of an equation; an important step when solving a cubic by Cardano's method, although Descartes does not mention this point here. Secondly negative roots can be eliminated by being rendered positive, though the benefit of this is not clear when, for instance, a solution to a particular problem is sought (p. 376) :

> 'Now this method of transforming the roots of an equation without determining their values yields two results which will prove useful: First we can always remove the second term of an equation by diminishing its true roots by the known quantity of the second term divided by the number of dimensions of the first term, if these two terms have opposite signs, or, if they have like signs, by increasing the roots by the same quantity.'

After two illustrative examples (p. 377), he continues :

> 'Second by increasing the roots by a quantity greater than any of the false roots we make all the roots true. When this is done, there will be no two consecutive + or - terms; and further, the known quantity of the third term will be greater than the square of half that of the second term. This can be done even when the false roots are unknown, since approximate values can always be obtained for them and the roots can then be increased by a quantity as large or larger than is required.'

Next, equations with irrational coefficients are dealt with, a problem seldom dealt with by mathematicians. A method for rationalising them by multiplication or division by successive powers of a suitable quantity is given, which can also be used to make any coefficient take a particular value. This is not the simplest way of achieving the latter.

Descartes next comes to complex roots, and mentions here that the manipulations described cannot render complex roots real. The implication is that the manipulations are all real (p. 380) :

> 'Neither the true nor the false roots are always real; sometimes they are imaginary; that is while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation $x^3 - 6x^2 + 13x - 10 = 0$ as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, always remain imaginary.'

Later, where Descartes is writing about quartics, there is an early example of separating a problem impossible of solution, from its algebraic equation with complex roots (p. 386) :

> 'Now these two equations have no roots either true or false, whence we know that the four roots of the original equation are imaginary; and that the problem whose solution depends on this equation is plane [constructable using compass and straight edge only] , but that its construction is impossible, because the given quantities cannot be united [combined in the same problem] .'

There are two further references to a connection between impossible constructions and complex roots (pp. 393,406), however he also says (p. 401) :

> 'I have not yet stated my grounds for daring to declare a thing possible or impossible'

So, although impossible constructions are associated with complex roots, Descartes is not saying that these are the only sources of such roots. Later mathematicians have found this association useful.

Descartes' view of complex roots is indicated where they are described as 'merely imaginary' (p. 400), and we know that he did not necessarily regard negatives as suitable to be the roots of quadratics from the earlier omission of a negative root (p. 302). The second remark quoted above (p. 386), states that an equation whose roots are neither positive nor negative has no roots, although the quartic they were derived from is said to have roots that are imaginary. Therefore 'imaginary' is here another word for non-existent.

Descartes' rule of signs says that the number of changes between + and - in an equation gives the number of positive roots, and the number of repeats of + or - gives the number of negative roots. Where the roots are real, this rule is entirely satisfactory, but where they are complex it continues to give numbers of 'positive' and 'negative' roots, without distinguishing between real and complex ones. Considering equations with positive real roots a and b :

	Changes	Repeats	Roots
$(x-a)(x-b) = x^2-(a+b)x + ab = 0$	2	0	both positive
$(x-a)(x+b) = x^2-(a-b)x - ab = 0$ $(x+a)(x-b) = x^2+(a-b)x - ab = 0$	1	1	one positive, one negative
$(x+a)(x+b) = x^2+(a+b)x + ab = 0$	0	2	both negative

Considering equations with complex roots :

$(x-(-1+i))(x-(-1-i)) = x^2+2x+2 = 0$	0	2	both negative ?
$(x-(1+i))(x-(1-i)) = x^2-2x+2 = 0$	2	0	both positive ?

(for clarity I have given specific examples in the complex cases)
An attempt to construct a quadratic equation with complex roots, one 'positive' and one 'negative', taking h and k positive, gives the choices: either $x^2 + hx - k = 0$, or $x^2 - hx - k = 0$. In each case there is one change of sign so, by the rule of signs, one root positive and one negative, and there is no other way of arranging the coefficients to obtain this result. However, in each case, the discriminant is positive so neither equation can have complex roots. It is clear that where an equation has roots of opposite sign, they cannot be complex. Inspection shows that the Descartes rule, applied to an equation with complex roots gives correctly the sign of the real part of the root. But the real parts must be equal so a quadratic with complex roots cannot have just one change of sign. The rule operates independently of the imaginary part of the roots giving the sign of the real part when there are no other factors. However, as Newton noticed later, the introduction of further factors (and so roots) can change the sign given to existing complex roots by the rule. Unfortunately this does not always happen so this interesting property cannot be used to detect complex roots.

Descartes gives no proof of his rule of signs for real roots and does not explore how it operates when the roots are complex.

Although Descartes considers irrational coefficients and accepts negative ones readily, he does not consider complex ones. It seems that the acceptability of different number categories varies partly with their context. He only considers irrational coefficients on one occasion, but many mathematicians never mention them at all.

A remark on page 400 shows that Descartes was familiar with the association between the cube of a quantity and the volume of a cube, although, in this case, it is the root of a cubic that is being described as the side of a given cube. Two solutions to a problem are given, in the first a geometrical construction is used and the answer given in terms of arc and chord lengths of a circle, in the second Cardano's method is applied to the appropriate cubic equation and the answer given as the sum of two cube roots. Descartes uses the square root sign where appropriate in the Geometry, but his remark about the cube root shows that he may have thought of the square root as the side of a square of given area. If this is so, it is difficult to see how Descartes could have acquired any very advanced ideas about complex numbers. He had eventually followed Girard in stating cautiously that an equation of degree n could have n roots, but gave no proof. Although this seems to mean that he was counting complex roots together with real ones, Descartes never fully accepted complex numbers as numbers and had doubts about negatives.

Sir Isaac Newton 1642-1727

Newton's interest in algebra seems to have started when, in about 1670, he came across Algebra Ofte Stel-konst by Gerhard Kinckhuysen, which members of the Royal Society were having translated from Dutch into Latin for publication in England(1). Newton revised the book and produced a commentary, but it never reached publication. Wallis also sent him an early draft of his Algebra for comment(2), and the published version of 1685 includes a number of Newton's ideas.

Newton is known to have read works by Descartes (La Géométrie), Wallis (Arithmetica Infinitorum), Heurat, de Witte, Hudde, Vieta, Oughtred (the Clavis), Huyghens, Fermat, Gregory and Barrow (Euclid's Elements, Euclid's Data). Of these, La Géométrie had the greatest influence on Newton in the area of algebra, and Euclid's work in that of proof.

In Observations on Kinckhuysen(3) Newton gives a method for obtaining the cube root of a complex quantity and uses it to evaluate the roots in the irreducible case of the cubic. In a letter of 1677, in reply to a query from Leibniz, Newton says (erroneously)(4) :

> 'A possible root is always expressed by a possible series, an impossible one always by an impossible'

By 'possible series' Newton means one that is convergent, an 'impossible series' is divergent. He associates a real root with a convergent series and a complex root with a divergent series, linking the impossibility of the root with the infinite nature of the sum of the series. No proof is given. Although Newton is content to manipulate complex numbers and describe some of their properties, he does not discuss their nature; the example given shows that he had some original ideas on this point, not necessarily well founded.

(1) Whiteside, Mathematical Papers of Isaac Newton, II, p. 280

(2) Whiteside, II, p. xiii

(3) Whiteside, II, pp. 377-95

(4) Whiteside, IV, p. 541

A rule for determining the number of complex roots in an equation was given by Newton in lectures given at Cambridge from 1673 to 1683[1]. The method, known as 'Newton's Rule', applies to polynomials of the form $f(x) = 0$, and is considerably more complicated than Descartes' rule. The equation is written in the form $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots = 0$. Above the term in x^{n-r} is written the fraction $\frac{n-r}{r+1} \cdot \frac{r}{n-r+1}$, and calling this '$b_r$', if $b_r \cdot a_r^2 > a_{r+1} \cdot a_{r-1}$ then underneath is written a '+' sign, otherwise a '−' sign is written. Then the number of changes of sign in this row gives the number of complex roots. Newton gave no proof of this rule, which was not verified until 1865 by Sylvester[2]. The development of Newton's thinking on this problem is seen in his manuscript "Of Equations" written in 1665-6, in which he says[3]:

> 'Thus the signes of this Eq: $x^3 - pxx + 3ppx - q^3 = 0$ shew it to have three true [positive] roots, wherefore if it bee multiplied by $x + 2a = 0$ the resulting equation
>
> $x^4 + px^3 + ppxx \; {}^{+6p^3}_{-q^3} \; - 2pq^3 = 0$ [4] should have three true roots and a false [negative] one, but the signes shew it to have three false and one true. I conclude therefore that the two roots w^ch^ in y^e^ one case appeare true, and in the other false are neither, but imaginary; and that of y^e^ other two roots, one is true y^e^ other false.'

Newton is taking p and q to be positive reals and finds that the introduction of another root $-2p$ has, in this case, changed the signs of two of the roots in the cubic. These roots he takes to be complex, as this could not have happened had all the roots been real. After remarking that finding the number of complex roots by such a rule could be more laborious than solving the equation, he gives these instructions[3]:

> 'Over y^e^ terms of y^e^ Equation set a series of fractions each having y^e^ dimensions of the terme under it for its numerator, & the number denominating y^e^ term first, second, third etc for its denominator. Then in every three terms observe whither the

(1) Mills, "The Controversy between Colin Maclaurin and George Campbell over Complex Roots", Arch. Hist. Ex. Sc., 28(1983), 149-64

(2) See Appendix I, p.(1)

(3) Whiteside, I, p.520-21, 526

(4) Correctly : $(x + 2p)(x^3 - px^2 + 3p^2x - q^3) = x^4 + px^3 + p^2x^2 + (6p^3 - q^3)x - 2pq^3$

square of the middle term multiplied by the fraction above be greater equall or lesse y^n y^e factus of the termes before & after it multiplied by y^e fraction over y^e terme before it. If greater write y^e sign + underneath; if equall or lesse write the sign - under=neath y^e middle terme: and lastly set + under y^e first terme of y^e equation. Then observe how many changes there are from + to - & conclude that there are soe many paires of imaginary roots. Unlesse all y^e roots bee equall.'

That this rule gives only a lower bound for the number of complex roots was known to Newton, as is indicated in the last sentence where he says[1] :

'Sometimes there may bee impossible roots not by this means discovered, w^{ch} if you suspect, augment or diminish y^e roots of the Equation a little, not soe much as to make them all affirmative or all negative, or at most not much more. & try the rule again. And if there bee any impossible roots twill rarely happen y^t they shall not bee discovered at two or three such tryalls. Nor can there bee an Equation whose impossible roots may not bee thus discovered.'

So a few 'tryalls' with a modified equation are necessary and, of course, the modifications must be real ones. Newton does not actually say that increases or decreases in the unknown are to be real, but would certainly have said so if this were not the intention; it is not possible to say whether a complex adjustment has augmented or diminished a number. A more important point is that Newton does not say how we are to know when sufficient trials have been made to find all the complex roots. The method must have been considered useful in spite of this defect, and as Newton says, it is unlikely that the number of complex roots could not be discovered quite quickly, particularly with equations of low degree.

In a lecture given in 1681, Newton returns to the problem of finding the number of complex roots in an equation without solving it[2]. He says that it is possible to find whether the complex roots are among the positive or the negative roots by examining the signs that have been written over the terms in the method for finding the total

(1) Whiteside, I, p.526

(2) Whiteside, V, p.351

number of complex roots. Then the number of 'positive' complex roots (that is with real part positive), is given by the number of changes in consecutive signs, and the number of 'negative' ones by the number of repeats. There is no proof of this very satisfactory discovery, but a number of examples is given.

An interesting graphical idea is described in a manuscript on the roots of equations, written in the late 1670's[1]. In this Newton says that complex roots may be represented by 'folds' in curves. These are described as dips towards the horizontal axis which are not sufficient for the curve to intersect the axis. He must have been considering the question of representation of complex numbers graphically in the Cartesian plane, itself a fairly new idea. This representation does not seem to have proved useful. Newton did not take the step of moving into a third dimension.

Newton's main contribution to complex number theory was his rule for the number of complex roots in an equation. His description of the properties of the discriminant of a quadratic (not discussed here) gave a useful method for testing whether real roots would be found. Neither the association of complex roots with divergent series nor the graphical representation were fruitful. Newton was probably as disturbed as Descartes to find that real operations on the unknown did not eliminate complex roots from an equation. In a letter to Collins of 1670[2] he says :

> '. . . equations, to what terms soever they are reduced, their real roots never become imaginary nor their imaginary roots real, though indeed their true roots may become false and false ones true.'

Newton was one of many mathematicians who regarded complex answers to algebraic equations as of use in demonstrating that a problem was unsolvable. He wrote in Universal Arithmetick[3] :

> 'But it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible, as if they were possible.'

(1) Whiteside, V, p.35

(2) Rigaud , Correspondences , II, p.307

(3) Newton , Universal Arithmetick (1728) , p.193

Newton regarded complex numbers as useful, although he made little use of them himself and they never became central to any of his main interests. Newton was a practical man and does not appear to have speculated about the nature of complex numbers. His weight on the side of acceptance of them as useful entities worthy of note, may perhaps be regarded as his most positive contribution to their advancing status.

John Wallis 1616 - 1703

John Wallis was educated at Cambridge, but spent most of his life as Savilian professor of geometry at Oxford. He was familiar with Greek mathematics, lectured on the books of Euclid, Archimedes and Apollonius, and gave a solution to a well-known locus problem posed by Pappus. Wallis's book on algebra, A Treatise of Algebra both Historical and Practical, was published in 1685, but had been drafted much earlier, an early version having been sent to Newton for his comments.

The book starts with a substantial section acknowledging work done in algebra by the 'Grecians', the Arabs and European mathematicians. He lists both names and book titles and we may assume that he had at least a good working knowledge of these works and had probably read many, if not most of them. Greeks mentioned include Euclid, Pappus, Archimedes, Apollonius, Diophantus and Ptolemy and acknowledgement is made of the translation and republishing of their work by Xylander, Bachet and Fermat. Although mentioning the Arabs, he does not name any but passes on to Regiomontanus, Stevin, Briggs and Napier. Next is a list of algebraic works by Pacioli, Pisanus, Scipio, Cardano, Tartaglia, Bombelli, Ramus, Clavius, Recorde, Vieta (Specious Arithmetick), Oughtred and Harriot but Descartes is not included. He summarises works of Oughtred and Harriot, Cavilieri's indivisibles, his own Arithmetica Infinitorum and the method of exhaustion which is its justification, and the work on negative and fractional indices of Newton. There is a strong historical sense of the way in which Wallis is building on the ideas of others.

Wallis championed the English mathematician Harriot, and it was a recurrent claim of his that Descartes took many of his innovative ideas from Harriot without acknowledgement. In Chapter XXXI of the Algebra, Wallis says[(1)]:

> 'He [Harriot] takes in also the Negative or Privative Roots which by some are neglected. Wherein he is followed by Des Cartes save that what Harriot calls (very properly) Privative Roots, Des Cartes (I know not for what reason) is pleased to call False Roots.'

(1) Wallis, Algebra, p.128

This despite the fact that Harriot did not usually accept negative roots whereas Descartes was able to do so. Wallis would admit only Descartes' rule of signs as his sole innovation. Collins, in his correspondence with Wallis, tried to reconcile him to the originality of Descartes. It has been said that this blind spot of Wallis was because of an over-partiality for English mathematicians; however, in his Algebra, Wallis acknowledges many previous algebraists, rather few of whom were English and several of whom were French. Wallis's animosity may have been more personal. Apart from this one defect, Wallis can be seen as a bold and original thinker with a wide knowledge of the mathematical scene and with a strong historical sense of the development of mathematics and the main threads of mathematical thought.

Writing about some work of Harriot on the number of real roots in a polynomial, Wallis says in Chapter XL[(1)]:

> 'And having shewed it as to the Affirmative Roots, it may by like Methods, be shewed as to the Negative also: For (as was before shewed) by changing all the signs, those Negatives, will become Affirmatives, and the Affirmatives Negatives. So that what shall now be the Number and value of the Affirmatives, were before of the Negatives. Whereby it will appear how many in all be Real; and how many but Imaginary.'

Assuming that by 'real' and 'imaginary' Wallis means real and complex (and not positive and negative), he seems to be under the impression that taking the total number of roots and subtracting the positive and negative ones, gives the number of complex roots. He was aware of Descartes rule of signs, but seems not to know of Newton's rule for numbers of complex roots.

Before coming to the three contributions to complex number theory by Wallis which I consider to be of most importance, I should like to deal with some aspects of his thinking as shown in the Algebra. An interesting insight into his thinking about operations is given in Chapter I by his categorisation of them. He classifies addition, multiplication and 'constitution of powers' as synthetic operations or compositions; and subduction (subtraction), division and extraction of roots as analytic operations or resolutions. The distinction is that synthetic operations can always be performed but analytic ones are only sometimes possible, the former involving a building up and the

(1) Wallis, p.152

latter a breaking down. The three classified as analytic are those which lead to the introduction into the number system of, respectively, negative, fractional and complex numbers and this idea can readily be incorporated into a modern description of the complex number field. The analytic operations had long caused difficulties to mathematicians because the natural numbers are only closed under the synthetic operations. Wallis's point that synthetic operations only can always be performed shows that, to him, number still meant natural number.

Wallis's view of powers higher than three shows that Greek influence on him was strong. Although algebra can be applied to anything capable of proportion, he says later that there can be no power higher than three. From Chapter XXII (p.90) :

> '. . . algebra extends itself as far as Ration or Proportion may reach and therefore may be applied to anything that is capable of proportion. Line, surface, solid, time, weight, strength, number or whatever else may be esteemed to have Magnitude (as Euclide calls it), or Quantity (as we now use to speak).'

Referring to 'plano-plane' quantities, he says in Chapter XXX (p.126) :

> 'That is a Monster in Nature and less possible than a Chimaera or Centaure. For Length, Breadth and Thickness take up the whole of Space. Nor can we imagine how there should be a Fourth local Dimension beyond these Three. But if we consider a number . . .'

So there is no meaning to a fourth length dimension but if only numbers are being considered, there is no difficulty. However he does not restrict the quantities listed in Chap XXII to one dimension. Time, weight, surface etc raised to the second or third power would be just as much of a 'Chimaera' as length raised to the fourth power (more so), and there can be no meaning attached to a mixed sum of these quantities.

Wallis was well aware of the importance of notation and discusses it in some detail in the <u>Algebra</u> . He saw the proliferation of notations as a great handicap to algebraic development. For instance, he says next, (pp.91-2), that a fifth power of A could be written Aqc, AcAq, AqqA or AAAAA, where q indicates 'quadrato' and c 'cubo'. He says that qc would mean a fifth power to Diophantus, Vieta and Oughtred, but a sixth power to the Arabs, Pacioli, Stifel, Bombelli, Tartaglia, Cardano and Clavius. Wallis advocates the use of aaaa or a^4 instead of Aqq etc for a fourth power, and makes use of fractional and negative indices. He uses Descartes' index notation in spite of his strong antipathy to Descartes, but he did not justify the use of these, which was left to Newton. Wallis used $\sqrt{-1}$ for the square root of minus one rather than $(-1)^{\frac{1}{2}}$, and did not advocate the use of any single symbol for this.

Wallis had a high opinion of the Clavis of Oughtred, but considered it worth mentioning on several occasions in the Algebra that Oughtred omitted negative and complex roots. Wallis's own view of these seems to have been similar to that of Descartes in as far as he describes them both as 'impossible', that is to say, next to non-existent. Wallis, like Newton, sees their claim to consideration in their admitted usefulness. In Chapter LXVI about negative squares and their imaginary sides, he says (pp. 264-65) :

> 'These Imaginary quantities (as they are commonly called) arising from the Supposed Root of a Negative Square, (when they happen,) are reputed to imply that the Case proposed is Impossible.
>
> And so indeed it is, as to the first and strict notion of what is proposed. For it is not possible, that any Number (Negative or Affirmative) Multiplied into itself, can produce (for instance) -4. Since that Like Signs (whether + or -) will, produce + ; and therefore not -4.
>
> But it is also Impossible, that any Quantity (although not a Supposed Square) can be Negative. Since that it is not possible that any Magnitude can be less than Nothing, or any Number Fewer than None.
>
> Yet is not that Supposition (of Negative Quantities) either unuseful or Absurd : when it is rightly understood.'

Some imaginative and useful examples are then given, using, for instance, a man advancing and retreating for negative distances, and the sea advancing and retreating for negative areas.

Wallis rediscovered the so-called Cardano's method for the roots of a cubic and says in Chapters XXVIII and XXXVII (p.121, p.142) :

> 'I did before suspect that in superior equations, there might be more than two roots'
>
> '. . . how many Roots (Real or Imaginary,) every Equation contains (viz. so many as are the Dimensions of the Highest Term:)'

This was one of the first clear statements of the fundamental theorem of algebra, which was becoming more widely recognised. The methods for dealing with the irreducible case either by geometry or trigonometry were fairly widely known by this time, and Wallis says in Chapter XLVIII(1) :

> 'these equations which have been reputed desparate, are as truly solved as the others'

The desparation was caused by the fact that, where the roots are all real, two of them are given by an expression containing the cube roots of complex numbers. In this remark Wallis seems to be taking a very positive attitude to complex numbers where they arise in an intermediate step.

The correspondence between Wallis, Collins, Gregory and Newton took place from about 1673 to 1675 during the period when Wallis was drafting his Algebra . Among the topics covered were the solution of polynomials and the nature of roots, notations and meanings of complex roots, all of which were included in the Algebra . In a letter to Collins of 1673 about Cardano's rule, Wallis describes the imaginary parts as 'extinguished' by addition. The fact that these awkward quantities can be described as extinguished relieves the mathematician of the obligation to define them. In the same letter he says that all cubics are susceptible to Cardano's rule as(2) :

> '. . . the impossibility of [the square roots of negative quantities] hinders it not, unless where the binomial cube will not admit of an extraction of its root '

The second part of this remark seems to contradict the first as the rule always produces an answer, and by admitting complex numbers the roots can always be extracted. Although Wallis may have been thinking of the difficulties if irrationals are involved, he seems to be exhibiting a very ambivalent attitude to complex numbers; they produce impossibilities at the same time as being entirely acceptable. This appears to be a lapse of logic although it is not entirely clear what is meant by 'binomial cube'.

It was in another letter to Collins of 1673(3) that Wallis first gave one of his most important contributions to complex number theory. This is that the square root of a negative quantity (impossible as both of these are) can be regarded arithmetically as a mean proportional between a positive and a negative. He is still thinking in terms of

(1) Wallis, Algebra, p.181

(2) Rigaud, Correspondences , II, p.559

(3) Rigaud, II, p.576

length and areas and says :

> '. . . a negative plane may as well be admitted in algebra as a negative length, both being in nature equally impossible . . . and if we suppose such a negative square, we may as well suppose it to have a side, not indeed an affirmative or negative length, but a supposed mean proportional between a negative and positive thus designable $\sqrt{-n}$ or rather $\sqrt{-n^2}$, that is $\sqrt{(+n \times -n)}$ a mean proportional between +n and -n.'

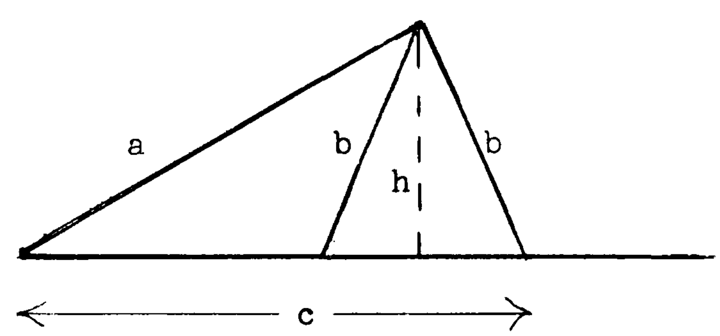
In the same letter he introduces rather diffidently the use of $2\sqrt{3}$ for $\sqrt{12}$ etc which he found most useful. In the Algebra Wallis seems to be thinking of the symbol '$\sqrt{}$' almost as an operator, in the sense that did not become widespread until the 19th Century. In Chapter LXVI he writes of $\sqrt{-1}$ (p.266) :

> '. . . $\sqrt{}$ implies a mean proportional between a Positive and a Negative Quantity. For as $\sqrt{bc}$ signifies a Mean Proportional between +b and +c; or between -b and -c; . . . . So doth $\sqrt{-bc}$ signify a Mean Proportional between +b and -c or between -b and +c; either of which being Multiplied, makes -bc. And this as to Algebraick consideration; is the true notion of such Imaginary Root, $\sqrt{-bc}$'.

Wallis later gives the example $\sqrt{2}$ as a mean proportional between 1 and 2. The use of mean proportionals was well accepted, being firmly founded in Euclid's Elements , and its introduction into complex number theory should have produced a considerable increase in their acceptability.

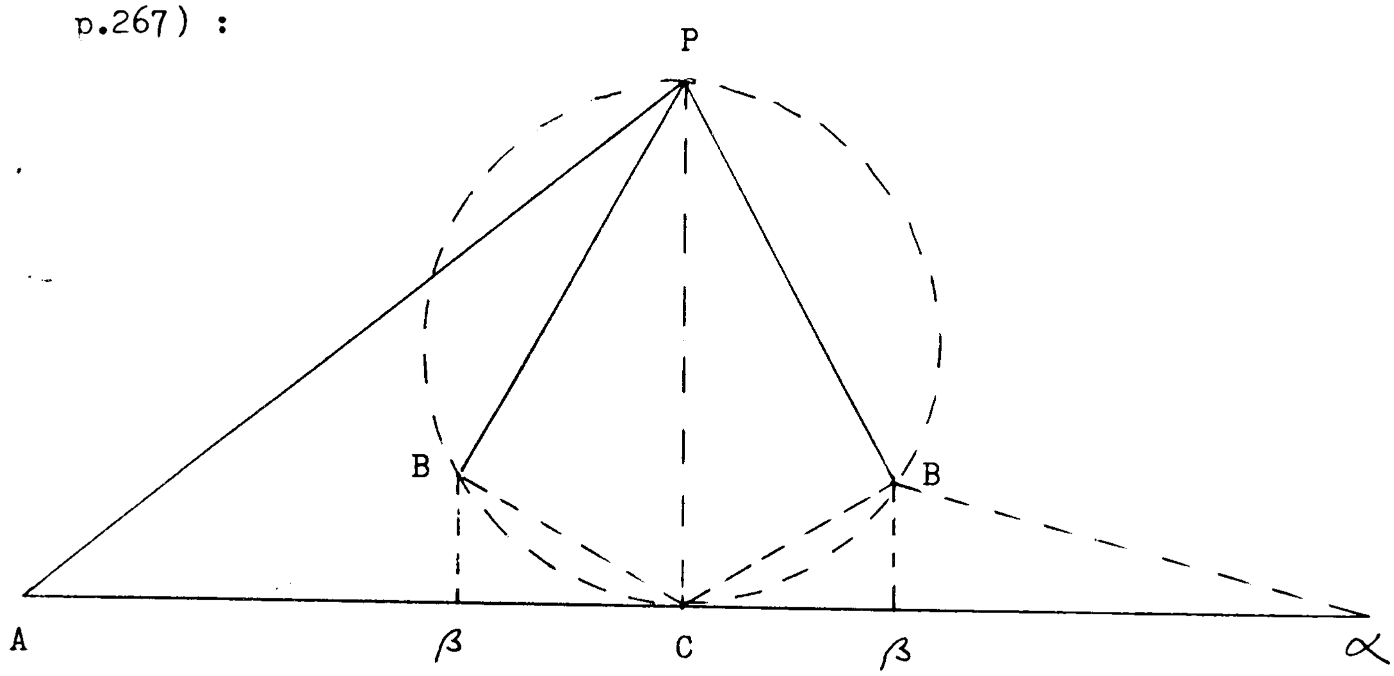
Wallis's second important contribution was his suggestion that an imaginary number can be 'found' not on the real number line, but 'above' it. He first uses a triangle problem to establish that square roots may be taken as either positive or negative.

He considers the ambiguous case in which it is required to solve a triangle given two sides and the included altitude (see figure)(1). a, b and h are known and c is obtained from two applications of



(1) Wallis, Algebra , p.266 (adapted)

Pythagoras' theorem : $c = \sqrt{(a^2 - h^2)} + \sqrt{(b^2 - h^2)}$. However, taking the second square root as negative gives a solution which is equally valid. He then gives the following example (see figure taken from p.267) :



With AP = 20, PB = 12 and PC = 15, Wallis says that AC = $\sqrt{175}$ and CB = $\sqrt{(144-225)} = \sqrt{-81}$. The diagram cannot be drawn with B on AC as required, but it is possible if B is placed above the line as in the diagram. The description is not that of Argand; Wallis is thinking in Euclidean terms and not in terms of Cartesian coordinates or rotations. However he clearly describes an arrangement in which negatives and positives lie on a line, and an imaginary is placed off the line. He says in Chapter LXVII (p.267-68) :

> 'Yet are there Two Points designed (out of that Line, but) in the same Plain; to either of which; if we draw the Lines AB, BP, we have a Triangle; . . . as were required: . . . .
> The greatest difference is this; That in the first Case, the Points B, B, lying in the Line AC, the Lines AB, AB, are the same with their Ground-Lines, but not so in this last case, where BB are so raised above ββ (the respective Points in their Ground-Lines, over which they stand,) as to make the case feasible;. . So that, whereas in case of Negative Roots, we are to say, The Point B cannot be found, so as is supposed in AC forward, but Backward from A it may in the same Line : we must here say, in case of a Negative Square, the Point B cannot be found so as was supposed, in the Line AC; but Above that Line it may in the same Plain.'

Wallis goes on to indicate that he sees this kind of representation as one for complex roots; it is only as roots of equations that he would have encountered such numbers. There is no question of any general concept of a number plane or full number status for complex numbers. He says[(1)] :

> 'What has been already said of $\sqrt{-bc}$ in Algebra, (as a Mean Proportional between a Positive and a Negative Quantity:) may be thus Exemplified in Geometry . . . .'
>
> 'This I have the more largely insisted on, because the Notion (I think) is new; and this, the plainest Declaration that at present I can think of, to explicate what we commonly call the Imaginary Roots of Quadratick Equations. For such are these.'

Wallis is trying to give a geometrical interpretation to complex numbers which will parallel the arithmetical mean proportional one.

Wallis goes even further. Several mathematicians had said that complex numbers are useful to indicate an unsolvable problem, both Wallis and Newton held this view. Collins had written to Wallis in an undated letter, that complex roots should be given as well as positive and negative ones[(2)] :

> '. . . their use being to shew how much the data must be mended to make the roots possible, and give points or bounds in delineations, shewing how much a curve must pass beneath or beyond a given right line, by aid whereof the roots are found'

Collins attributes the idea to 'Dr Pell'. Wallis tries to pursue this idea a little further by showing how the degree of impossibility might be quantified. He says of the distance βB (see diagram on previous page)[(3)] :

> 'This Construction shows that Case (so understood) to be Impossible; but how it may be qualified, so as to become possible.'

So the distance βB can be used to discover how the problem must be adjusted to make it solvable. He says in Chapter LXVIII, of complex roots[(4)]:

(1) Wallis , Algebra , p.268

(2) Rigaud , Correspondences II , p.481

(3) Wallis , Algebra , p.269

(4) Wallis , p.272

> '. . . which beside declaring the case in Rigour to be impossible, shew the measure of the impossibility; which if removed, the case will become possible. And they direct to such succedaneous operations in lieu of what is proposed, as may afford useful discoveries of somewhat which at the first Proposal was not thought of.'

Wallis is not able to give a quantitative relationship between the complexness of the roots and the impossibility of the problem in any particular case. He passes on the interesting idea of Pell and hints that even more useful discoveries may be made.

Wallis's third achievement is closely linked with the second. He gives a picture of the nature of complex numbers by producing a set of problems (mostly geometrical), that are impossible of solution. When treated algebraically they lead to complex roots. A rationalisation is given in each case, usually in terms of a geometrical adjustment in the problem. The effect is to show how geometrical alterations can render the answer to a problem possible or impossible. This is the closest Wallis comes to a concrete interpretation for a complex quantity. In one problem an attempt is made to find the third side of a right-angled triangle having misunderstood which is the right angle, in another a construction which only applies to a point in a line is used for a point not in the line. All these lead to the necessity to construct complex lengths. Wallis says in Chapter LXVIII (p.272) :

> 'The solution amounts to this: that the case proposed, as to the rigour of it, is impossible: But with such mitigations, it may be thus, and thus constructed.'

Among others, he lists the following faults which may need correction, some of which are equivalent to each other (pp.272-73) :

Use of

1. points not on the line proposed
2. tangent and secant instead of sine and cosine
3. a point above instead of in the line proposed
4. an inclined instead of a horizontal plane
5. an ellipse instead of a circle
6. a hyperbola instead of a circle
7. incorrect signs

In the chapters on the relationship between geometric problems and algebra, Wallis refers to the problems rather than the solutions as 'impossible'. In the preface and elsewhere, he refers to the '<u>Imaginary Roots</u> of Impossible Equations'(1). Unfortunately he is not consistent in his nomenclature, sometimes referring to complex numbers as 'impossible' and sometimes as 'imaginary'. There is an adverse side to the proposition that complex roots correspond to impossible problems. It emphasises the impossibility of giving an actual solution in these cases and so adds weight to the suitability of the word 'impossible' to describe the numerical answers.

The new arithmetical and geometrical interpretations of complex numbers put forward by Wallis represent a valuable contribution to their study, and show his skill in tackling an obscure and difficult concept. His ideas, well founded in Greek methods, should have enabled mathematicians to take up and use complex numbers with increased confidence. However, although Wallis's work was widely read, the next hundred years saw little change in their acceptability.

(1) Wallis, p.[v], Preface

Chapter III

The Algebra (1685) of John Wallis to the Algebra (1769) of Leonhard Euler

The period from the Algebra of Wallis to the Algebra of Euler saw great strides made in the results obtained from manipulating complex numbers, but no increase in insight into their nature. As the insights of Wallis do not appear to have had an influence on later mathematicians, it may be said that progress in this direction was retrograde. Leibniz said in 1702 that complex numbers are 'a fine and wonderful refuge of the divine spirit - almost an amphibian between being and non-being', and Euler, in more prosaic terms, gave his reasons for considering them impossible (see below). But by Euler's time, connections had been made between complex numbers and logarithmic, trigonometric and exponential functions. Cotes had the result $i\phi = \ln(\cos\phi + i\sin\phi)$ in 1714, and de Moivre's theorem was essentially known to him by 1722. Implicit in Cotes' result is the relationship $e^{i\Pi} + 1 = 0$, actually due to Euler, which can be regarded as the crowning achievement of the 18th Century, connecting as it does five fundamentally important natural, imaginary and transcendental quantities (one newly discovered), and the two operations addition and multiplication by means of the equality relation. These connections all became explicit after the discovery of the inverse relationship between exponential and logarithmic functions (published by William Jones in 1742) (1). By 1743 Euler had the formulae $\cos s = \frac{e^{is} + e^{-is}}{2}$ and $\sin s = \frac{e^{is} - e^{-is}}{2i}$ and in 1749 he wrote an article on the controversy between Leibniz and Jean Bernoulli about the logarithms of negative and complex quantities (2). In 1747 d'Alembert gave the first demonstration that all algebraic operations on complex numbers, including raising to powers, gave complex numbers of the form a + ib and not a hierarchy of new number species, as had been feared (see below).

(1) Kline, p.258

(2) Kline, p.409

An important history of mathematics produced in France during this period was <u>Histoire des Mathématiques</u> by Étienne Montucla, published in 1758. It is detailed and scholarly and was widely known. Montucla does not dwell long on complex numbers, but gives the properties of conjugates in connection with those of the roots of quadratic equations. He uses the word 'imaginaire' to mean 'imaginary'(1). There is no description of their general properties, no estimate of their usefulness and no discussion of their nature.

Nicholas Saunderson 1682-1739 , Abraham de Moivre 1667-1754

A mathematician working in England in the early 18th Century was Nicholas Saunderson. He lost his sight as a baby and later learned mathematics from hearing the works of Euclid, Archimedes and Diophantus read to him in Greek. He went to Cambridge and in 1711 followed Newton as Lucasian professor, later becoming, like Newton, a Fellow of the Royal Society. Saunderson's text-book <u>The Elements of Algebra in ten books</u> , was published in 1740, in two volumes. In it are covered such topics as the arithmetic of negatives, square roots of fractions, quadratic equations, indices, Newton's binomial theorem, logarithms, surds and the theory of equations. His method of obtaining the rules for multiplication of negatives with positives or negatives uses arithmetic progressions to get the correct results. All sections include many worked examples with detailed explanations. Complex numbers are dealt with a number of times, and are referred to as 'impossible'. It is clear that Saunderson does not think of them as numbers on a par with reals and that he thought of them not only in algebraic, but apparently also in near spatial terms, which is quite unexpected. Saunderson devised a system as an aid to the blind which he called

(1) Montucla, <u>Histoire</u> , p.80

'palpable arithmetic', in which he used a pin-board to represent certain arithmetical ideas. Apparently he did not extend this to cover complex quantities.

Saunderson's view of complex numbers was that they are impossible in the non-existent sense, but can be treated mathematically. He says in the <u>Algebra</u> (1) :

> '. . . -16 is no square number, since there is no root either affirmative or negative, which multiplied into itself will produce -16'

and later (p.184) :

> '$\sqrt{(-2)}$ is not only an inexpressible quantity but also an impossible one; and consequently . . . the two values of x in this equation $\left[x^2 - 4x + 6 = 0\right]$ $2+\sqrt{(-2)}$ and $2-\sqrt{(-2)}$ will both be impossible.
> N.B. Though the roots of this last equation be impossible in their own natures, yet they may be abstractly demonstrated to be just . . . by making $s = \sqrt{(-2)}$ and consequently $ss = -2$.'

On the same page, Saunderson describes complex roots in terms of a limit. Roots pass between real and complex via a limiting value where they are equal (p.184) :

> '. . . it appears that one root of a quadratic can never be impossible alone, but that they must either be both possible or both impossible : for . . . the impossibility of the roots flows from the impossibility of the quantity s or of the square root of ss when it is negative . . . the two unequal roots of a quadratic equation grow nearer and nearer to a state of equality as they grow nearer and nearer to a state of impossibility but do not come to be equal till they cease to be real, or at least, till they come to the limit between possibility and impossibility.'

He is thinking in dynamical, almost visual terms, and there are interesting implications for graphical representation in his remarks about the roots verging together. This kind of thinking was not usual at the time, although it is reminiscent of Newton's idea for graphical representation of complex roots (see above). Saunderson shows a very clear understanding of the behaviour of the roots of a quadratic.

That Saunderson's clarity of thought extends beyond the second degree is shown in some later remarks. He does not quite give the fundamental theorem of algebra in the terms of Gauss, and there is no proof, but he comes near to this later when he writes (p.679) :

(1) Saunderson, <u>Algebra</u> , p.83

'. . . in every equation the number of impossible roots is always even because the roots of a quadratic equation must always be both possible or both impossble . . . but if the index of the highest term of an equation be an odd number, it must have at least one root possible.'

Again Saunderson gives no proof of these observations, but they illustrate how advanced were many of his ideas.

Appended to Saunderson's book is a letter from Abraham de Moivre in answer to a query from Saunderson about the cube root of a complex number. He uses a primitive version of what is now known as 'de Moivre's theorem' in which an expression involving trigonometric functions is substituted for the quantity whose root is required. Saunderson has expressed himself as dissatisfied with Wallis's method, based on Cardano's method, which de Moivre describes as merely a trial. De Moivre finds the cube root by cubing an assumed root and equating real and imaginary parts of this with those of the original. There is no diagram, but the equation is expressed in terms of the sine and cosine of an angle. The cube root can be found by dividing the angle by three. de Moivre says[1] :

> '. . . if the original equation had been such as to have its roots irrational, his trial would never have succeeded. But farther I shall prove, that the extracting the cube root . . . is of the same degree of difficulty as that of extracting the root of the original equation . . . and that both require the trisection of an angle for a perfect solution.'

There follows an explanation with examples, running to about three pages.

(1) Saunderson , p. 745

Colin Maclaurin 1698–1746

The able Scottish mathematician Colin Maclaurin, who spent most of his working life as Professor of Mathematics at Edinburgh University, took up some of Newton's ideas, in particular the determination of the number of roots of a polynomial. In a letter to Stirling of 1728 he stated that if a polynomial of degree n has at least one pair of complex roots, then so has the quadratic obtained from its (n-2)th derivative.(1) This led to a plagiarism controversy with George Campbell who was also working on the number of complex roots of a polynomial. Campbell had noticed that if a polynomial has only real roots, then so will its derivatives, but he did not then infer Maclaurin's result about the number of complex roots. Maclaurin showed that Newton's rule is not reliable in the sense that it does not infallibly detect complex roots, although Newton himself knew this. It gives only a lower bound for these roots. He considered the polynomial $x^5 - 10x^4 + 30x^3 - 44x^2 + 32x - 9 = 0$ which has complex roots, which Maclaurin says are not detected by Newton's rule. However, applying Maclaurin's differentiation method, the complex roots are still not found as with $f(x) = x^5 - 10x^4 + 30x^3 - 44x^2 + 32x - 9$, $f'''(x) = 60x^2 - 240x + 180$ and $f'''(x) = 0$ has two real roots. This method also only gives a lower bound for the number of complex roots. The polynomial actually has three real roots, two between 0 and 1 and one between 6 and 7, and two complex ones.

Maclaurin's A Treatise of Algebra in Three Parts , an algebra text-book, was published posthumously in 1748, although he had intended to publish it many years earlier. It was assembled from his notes, with direct quotations from Maclaurin's writings given in double quotation marks. It is mainly concerned with the roots and general properties of polynomials and the inter-relation between algebra and geometry. His explanation of what is meant by an 'imaginary' quantity

(1) Mills , "The Controversy . . . ", Arch. Hist. Ex. Sc., 28(1983), 149–64

is given and complex quantities used freely where appropriate. There is no discussion of a philosophical nature about $\sqrt{-1}$ or complex numbers, but his views can be inferred even though Maclaurin did not write the book himself.

Maclaurin thinks of imaginary quantities as non-existent, he says that quadratics sometimes cannot be solved, but does not distinguish between a problem and its algebraic solution. From the Algebra, Part I, Chapter 13[1] :

> 'Since the squares of all quantities are positive, it is plain that "The square root of a negative quantity is imaginary, and cannot be assigned." Therefore there are some quadratic equations that cannot have any solution.'

The word 'impossible' is not used here but does occur later in the book.

He gives a rather free definition of rationals and irrationals using commensurability, defining them relative to an unspecified starting quantity. From Chapter 14 (p.95) :

> '. . . if any one quantity be called rational, all others that have any common measure with it, are also called rational: But those that have no common measure with it, are called irrational quantities.'

It may be said that what is being defined is relative rationality and relative irrationality. Maclaurin shows that the square roots of naturals (with the exception of 1, 4, 9 . . .) are incommensurable with naturals, and gives the method for rationalising denominators of the form $\sqrt{5} - \sqrt{3}$.

In a supplement to Chapter 14, Maclaurin shows how to find algebraically the cube root of a quantity of the form $a + b\sqrt{-1}$, which arises in Cardano's method, and also obtains the cube roots of 1. However, he does not consider this the best way and reiterates Saunderson's view that de Moivre's trigonometric method is to be preferred. De Moivre's theorem must indeed have seemed an elegant way of overcoming the difficulty. Maclaurin says (p.127) :

(1) Maclaurin, Algebra (1748), Part I, p.87

'But for a general and elegant solution, recourse must be had to Mr. de Moivre's Appendix to Dr. Saunderson's Algebra . . . what has been explained above may serve, for the present, to give the Learner some notion of the composition and resolution of those cubes; that he need not hereafter be surprised to meet with expressions of real quantities which involve imaginary roots.'

Maclaurin points out the fact that the resolution of the irreducible case shows that real quantities may be expressed using imaginary elements. The implication of accepting this is to raise the acceptability of complex numbers.

Maclaurin takes a broad view of polynomials, considering in Part II, those with either coefficients or powers that are fractional or irrational. The numbers of positive and negative roots of a polynomial are discussed, and the limits of their values. The rule given for the number of complex roots is the same as Newton's. No proof is attempted, reliance being placed on careful instructions and many examples. From Part II Chapter 11 (1) :

'The number of impossible roots in an equation may, for most part, be found by this

RULE

"Write down a series of fractions whose denominators are the numbers in this progression 1, 2, 3, 4, 5 etc continued to the number which expresses the dimension of the equation. Divide every fraction in the series by that which precedes it, and place the quotients in order over the middle terms of the equation. And if the square of any term multiplied into the fraction that stands over it gives a product greater than the rectangle of the two adjacent terms, write under the term the sign +, but if the product is not greater than the rectangle, write -; and the signs under the extreme terms being +, there will be as many imaginary roots as there are changes of the signs from + to -, and from - to +.

Thus the given equation being $x^3 + px^2 + 3p^2x - q = 0$, I divide the second fraction of the series $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$ by the first, and the third by the second, and place the quotients $\frac{1}{3}$ and $\frac{1}{3}$ over the middle terms in this manner,

$$\begin{array}{cccc} & \frac{1}{3} & \frac{1}{3} & \\ x^3 + & px^2 + & 3p^2x & - q = 0 . \\ + & - & + & + \end{array}$$

Then because the square of the second term multiplied into the fraction that stands over it, that is $\frac{1}{3}.p^2x^4$ is less than $3p^2x^4$ the rectangle under the first and third terms, I place under the second term the sign - : but as $\frac{1}{3}.9p^4x^2$ $(= 3p^4x^2)$ the square of the third term multiplied into its fraction is greater than nothing, and consequently much greater than $-pqx^2$ the negative

(1) Maclaurin , II , pp. 275-79

product of the adjoining terms, I write + likewise under x^3 and $-q$ the first and last terms; and finding in the signs thus marked two changes, one from + to −, and the other from − to +, I conclude the equation has two impossible roots. In like manner the equation $x^3 - 4x^2 - 4x - 6 = 0$ has two impossible roots;

$\quad\quad\frac{1}{3}\quad\frac{1}{3}$

$x^3 - 4x^2 - 4x - 6 = 0$ and the equation $x^4 * - 6x^2 - 3x - 2 = 0$

\+ \+ − \+ $\quad\quad\frac{3}{8}\quad\frac{4}{9}\quad\frac{3}{8}$

the same number $x^4 * - 6x^2 - 3x - 2 = 0$ . . . "'

\+ \+ \+ − \+

There follow examples, some with terms missing, up to the seventh power. Next, by comparison of the signs of the coefficients with those written below, the method given by Newton, Maclaurin shows how to determine whether the complex roots are among the positive or negative ones. He does not mention the difficulty observed by Newton, that the introduction of certain factors changes the signs of complex roots.

Maclaurin concludes this section (1) :

> '"This always holds good unless, which sometimes may happen, that there are more impossible roots in the equation than are discoverable by the Rule." This rule hath been investigated by several eminent Mathematicians in various ways; and others, similar to it, invented and published . . .'

He is aware of the fact that this procedure is not entirely rigorous.

Maclaurin reiterates some of the ideas of Wallis about the relationship between geometrical problems and their algebraic solutions. He makes the point that there are problems whose solution cannot be illustrated geometrically for which algebraic solutions of a sort can be given. From Part III Chapter 1 (2) :

> 'In Algebra, the root of an equation, when it is an impossible quantity, has its expression; but in Geometry, it has none. In Algebra you obtain a general solution, and there is an expression, in all cases, of the thing required; only, within certain bounds, that expression represents an imaginary quantity, or rather, "is the symbol of an operation which, in that case, cannot be performed;" and serves only to show the genesis of the quantity, and the limits within which it is possible.
>
> In the geometrical resolution of a question, the thing required is exhibited only in those cases when the question admits of a real solution; and beyond those limits, no solution appears.'

(1) Maclaurin , II , p.279

(2) Maclaurin , III , p.299

This is followed by a number of examples similar to those given by Wallis in his Algebra . For instance, it is possible to calculate a value for the distance from the centre of a circle to its intersections with a straight line, but a drawing can only be made when this distance is less than the radius. Thus, in spite of the entrenched position of geometry, algebra is more powerful.

It is not possible to infer from this book alone, without Maclaurin's own notes, his usage of the words 'impossible' and 'imaginary'. In the book no distinction is made between their applications to problems or to numbers, but in the quotations from his actual writings both words are used to refer to solutions of equations. The importance of Cardano's method for acceptance of complex numbers comes out clearly in the book, and also the impetus from the algebra of polynomials towards a closer understanding of the number system. In the section on the links between algebra and geometry complex numbers are treated as quantities obeying certain rules and no observations made about their nature, although this would be the natural place for them.

It is noticeable, particularly where he gives his 'Rule', that Maclaurin takes a narrow view of unspecified numbers. Although at various points he considers irrational and fractional coefficients and powers, undoubtedly his coefficients 'p' and 'q' are positive, and possibly also naturals. He does not assume the freedom to assign to p and q any number value. An irrational coefficient, for instance, is specified if intended. Unfortunately this restriction weakens Maclaurin's demonstration of the rule's validity. He does not consider complex numbers as coefficients or powers. He is one of many mathematicians who have been obliged to widen their view of numbers that can be roots, without allowing this wider view to percolate to numbers in certain other situations.

Jean d'Alembert 1717-1783

D'Alembert was a mathematician working in France during this period. He was interested in developing mathematical techniques that could be applied to particular problems, and was able to make an important contribution to complex number theory. He became science editor to the Encyclopédie under the principal editorship of Denis Diderot, and contributed a number of articles. Although he was familiar with current ideas about complex numbers, he made only brief reference to them in this work(1). His prize-winning essay Réflexions sur la cause générale des Vents was published in 1747 and in it d'Alembert gave the first demonstration that a complex number raised to a complex power produced another complex number. His main difficulty was to determine a value for $\sqrt{-1}^{\sqrt{-1}}$; the method used was to consider variations in the real and imaginary parts of a complex function. The following is taken from Article 79 (2) :

> 'Car il est certain qu'une quantité, algébrique quelconque, composée de tant d'imaginaires qu'on voudra, peut toujours se réduire à $A+B\sqrt{-1}$, A & B étant des quantités réelles; d'où il s'ensuit, que si la quantité proposée doit être réelle, on aura $B = 0$.
>
> Pour démonstrer cette verité, il faut remarquer,
>
> 1°. Que $\frac{a+b\sqrt{-1}}{g+h\sqrt{-1}} = A+B\sqrt{-1}$, puisque $a = gA-hB$; $b= Ah+gB$; d'où l'on tire $A = \frac{bh+ag}{hh+gg}$; et $B = \frac{bg-ah}{hh+gg}$.
>
> 2°. Que $\left[a + b\sqrt{-1}\right]^{g+h\sqrt{-1}} = A + B\sqrt{-1}$. Car faisant varier A & B, aussi-bien que a & b, et prenant les différentielles Logarithmiques, on a $(g + h)\sqrt{-1}^{*} \times \frac{da + db\sqrt{-1}}{a + b\sqrt{-1}} = \frac{dA + dB\sqrt{-1}}{A + B\sqrt{-1}}$;
> c'est-à-dire
>
> $$\frac{AdA+BdB+(AdB-BdA)\sqrt{-1}}{AA + BB} = \frac{gada+gbdb-ahdb+bhda}{aa + bb} + \frac{(hada+hbdb+gadb-gbda)\sqrt{-}}{aa + bb}$$
>
> donc $AA + BB = \left[aa + bb\right]^{g} \times c^{-h\int\frac{adb-bda}{aa+bb}}$

(1) Diderot and d'Alembert , Encyclopédie VIII , p.560

(2) d'Alembert , Réflexions sur la cause générale des Vents , p.141

* Correctly : $(g + h\sqrt{-1})$

et $\int \frac{AdB - BdA}{AA + BB} = h \log \sqrt{\left[aa+bb\right]} + g \int \frac{adb-bda}{aa+bb}$.

Or $\int \frac{adb - bda}{aa + bb}$, et $\int \frac{AdB - BdA}{AA + BB}$ sont des expressions des angles dont les tangentes sont $\frac{b}{a}$ et $\frac{B}{A}$: donc B et A sont les sinus et cosinus d'un angle dont le rayon est

$\sqrt{\left[\frac{\quad}{aa + bb}\, g \times c^{-h\int\frac{adb-bda}{aa+bb}}\right]}$, et dont la valeur est

$h \log \sqrt{\left[aa+ bb\right]} + g\int \frac{adb - bda}{aa + bb}$.

3°. Il est évident, que $a+b\sqrt{-1} \pm (g+h\sqrt{-1}) = A + B\sqrt{-1}$; et que $(a + b\sqrt{-1}) \times (g + h\sqrt{-1}) = A + B\sqrt{-1}$.

4°. Par le moyen de ces trois propositions, il sera facile de réduire toujours à la forme $A + B\sqrt{-1}$, une quantité composée de tant et de telles fortes d'imaginaires q'on voudra. Car en allant de la droite vers la gauche, on sera évanouir l'une après l'autre toutes les quantités imaginaires, excepté une seule : la quantité proposée se reduire donc à $A + B\sqrt{-1}$; et si elle doit être une quantité réelle, B sera necessairement = 0.'

D'Alembert dismisses the sum, difference and product of two complex numbers in his third point and deals briefly with the quotient in point one. In point two he considers raising a complex number to a complex power. The method is to assume the result he is seeking, that it is of the form $A + B\sqrt{-1}$, and show that this assumption leads to no contradiction. The first step is to apply logarithmic differentiation. The use of the letters a, b, A and B as variables makes his work difficult to follow, and he uses differentials throughout. He treats $\sqrt{-1}$ as a constant; for instance, in $d(b\sqrt{-1})$, written $db\sqrt{-1}$, $\sqrt{-1}$ is removed in the next step as a factor. After differentiation, he separates real and imaginary parts to obtain two differential equations which are solved by standard methods using an integrating factor. He shows that A and B are the sine and cosine of the same angle, so there is no inconsistency in the original assumption. Other mathematicians, notably Euler, later supplied more rigorous proofs.

A value for $\sqrt{-1}^{\sqrt{-1}}$ may be determined more easily using the relationship $e^{i\theta} = \cos\theta + i\sin\theta$, with θ put equal to $\Pi/2$, say. This gives $e^{i\Pi/2} = i$, the ith power of which gives $i^i = e^{-\Pi/2}$, approximately 0·208 . . . , a real number. The general result $(a + ib)^{g + ih} = A + iB$ can readily be obtained using de Moivre's theorem and the relationships $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$. These relationships were known by the time of d'Alembert. Cotes had given the equivalent of $i\theta = \ln(\cos\theta + i\sin\theta)$ in 1714 and Euler knew that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ by 1740. In 1743 he published this discovery with the corresponding result for $\sin\theta$, and later rediscovered Cotes' result[1].

D'Alembert later, in connection with his work of 1752 on fluids, took the first steps in complex function theory. He found that $\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}$ and $\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}$ (now known as the Cauchy-Riemann equations), define two functions p and q, such that $dq = Mdx + Ndy$ and $dp = Ndx - Mdy$. qdx and pdy are exact differentials with p and q the real and imaginary parts of a complex function[2]. Euler later developed this method to evaluate real integrals using complex functions. The details of complex function theory are outside the scope of this thesis.

(1) Kline , p.409

(2) Kline , pp.626-27

Leonhard Euler 1707-1783

The Swiss mathematician Leonhard Euler, a student under Jean Bernoulli at Basle University, spent much of his working life as professor in the Academy of Sciences of St Petersburg, with a period of twenty-five years at the Academy of Sciences of Berlin. From about 1766 he became increasingly, and then totally, blind. Through his pupil, secretary and friend Nicolaus Fuss (1755-1826), he was able to continue his prodigious mathematical output.

Euler had real difficulties with the nature of complex numbers; he was quite open about his strong feeling that he was failing to grasp or understand their essential nature. He was well able to manipulate them according to certain rules of behaviour, but his intellectual bewilderment about them is clearly expressed in his Algebra. He is trying to get a clear picture of what he is discussing and is not content with the formalist view that $\sqrt{-1}$ is an entity which obeys a given set of rules. This difficulty seems to have been of much more importance to Euler than to other mathematicians such as Wallis, who also made important discoveries about complex numbers. I have not found an instance of Wallis expressing anxiety over the nature of complex numbers, either in his Algebra or his correspondence. I have also not been able to find that Euler was aware of Wallis' ideas about mean proportionals or his diagrammatic representation for complex numbers. It is difficult to believe that he was unaware of this material as Wallis' book was published in 1685 in English, and in 1693 in Latin; Euler could read both English and Latin. If Euler had been aware of it, we must conjecture that he would have mentioned it at the point where he discusses the nature of complex numbers in his own book. Elsewhere Euler mentions some of Wallis's ideas relating to infinite series, so must have been familiar with Arithmetica Infinitorum.

Euler had, at one time, views on negative numbers similar to those of Wallis. He discussed the behaviour of certain functions and their expansions in series.[1] Expanding $(1 - x)^{-1}$ by the binomial

(1) Euler, Opera Omnia, Ser.I, Vol.XIV, p.591; Vol.X, pp.78-81

theorem gives $1 + x + x^2 + x^3 + \ldots$ and by putting

$x = -1$ he obtained $\frac{1}{2} = 1 - 1 + 1 - 1 + \ldots$ (a)

with $x = -2$, $\frac{1}{3} = 1 - 2 + 2^2 - 2^3 + \ldots$ (b)

and with $x = 2$, $-1 = 1 + 2 + 4 + 8 + \ldots$ (c)

From $(1 + x)^{-2}$, $\infty = 1 + 2 + 3 + 4 + \ldots$, with $x = -1$. (d)

Comparing the last two, Euler argued that (c) must be greater than (d) on the basis of a term by term comparison. Therefore -1 is greater than infinity. It might as easily be argued that (c) lacks terms that are present in (d) and so -1 is less than infinity. It is difficult to see how Euler could have accepted the result (c), especially when he later obtained the result $-1 = 1 + 1 + 2 + 3 + \ldots$. Euler did not realise the importance of convergence, although he was aware of the concept, and obtained many baffling results. He regarded infinity as a limiting value between positives and negatives, similar to zero. As with Wallis, the discontinuity was being dealt with inadequately and divergence ignored.

In Chapter V of his Algebra Euler covers series derived from fractions with more care. He still does not mention convergence, but pays close attention to the remainder after summing the first few terms of a series. In most cases these series are geometric progressions of increasing terms and the remainders are found by application of the formula, which is only acceptable when the terms are decreasing. Because of the way this is done, these results seem to confirm the strange results above. For instance, taking $-1 = 1 + 2 + 4 + 8 + 16 + 32 + 64$, the remainder is $128/(1-2)$ or -128. This gives a total of $127 - 128$, which is -1. Euler does not seem to have subscribed to Newton's mistaken idea which associated divergent series with complex sums, but his unusual view of negatives did not provide a sound foundation for a study of complex numbers.

Euler's Elements of Algebra (1769) was written after he became blind, with the help of a tailor's unnamed young apprentice, who was completely ignorant of mathematics but who had been recommended by Bernoulli[(1)]. It was written as a beginner's text-book and the student

(1) Euler, Algebra (1840), p.xix (Memoir by Francis Horner); Hutton, Dictionary (1796), I, p.451

learned algebra as he went along; this ensured that the treatment was clear and easy for a novice to follow. Attached are the "Critical and Historical notes of M. Nicholas Bernoulli to which are added the additions of M. de la Grange". After its original appearance in Russia, a German edition was published in 1770 and English translations in 1797 and 1840. It was very popular and was reprinted and reissued a great many times[(1)]. It is noteworthy that although complex numbers are frequently mentioned, and the remarks about the nature of $\sqrt{-1}$ much the most contentious part of the book, the "Additions" contain only a passing reference to the topic. Furthermore, neither Bernoulli nor Lagrange seem to have noticed the errors in Euler's text (see below), and these remained uncorrected in the 1840 English edition.

Euler tried to give a definition of complex numbers by starting with a global concept of number, then eliminating all those number categories which do not have their properties. If complex numbers are numbers, then whatever remains must constitute a definition, or at least a description, of them. Unfortunately, his initial concept of what constitutes a number was too narrow, only encompassing reals on a one-dimensional model, so that after the elimination process nothing remained. This is given as a reason for calling the numbers 'imaginary' and 'impossible'. This elimination process could have brought Euler close to a working definition of complex numbers if his starting premises had been wider, in this sense he came close to a useful definition. Euler says in paras 141 to 144 of the <u>Algebra</u>[(2)] :

> '. . . the root in question must belong to an entirely distinct species of number; since it cannot be ranked either among the positive or among negative numbers.
> . . . positive numbers are all greater than nothing, or nothing, and . . . negative numbers are all less than nothing or nothing; so that whatever exceeds nothing, is expressed by positive numbers, and whatever is less than nothing, is expressed by negative numbers. The square roots of negative numbers, therefore, are neither greater nor less than nothing. We cannot say however that they are nothing; for nothing multiplied by nothing produces nothing, and consequently does not give a negative number.
> . . . we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers

(1) It is shortly to be reissued (1985)

(2) Euler, <u>Algebra</u> (1797), p.64

> which from their nature are impossible. Those numbers are usually called imaginary quantities, because they exist merely in the imagination.'

Complex numbers have been excluded completely from the number system, and are 'merely' imaginary, or non-existent.

Euler then says that as we can imagine these numbers and describe their behaviour, we can make use of them. From para 145 (p.66) :

> '. . . nothing prevents us from making use of these imaginary numbers, and employing them in calculation.'

He expands on this point in para 151, where he stresses their usefulness in showing that a problem is impossible (p.68) :

> 'It remains for us to remove any doubt which may be entertained concerning the utility of the numbers of which we have been speaking; for those numbers being impossible it would not be surprising if they were thought entirely useless, and the object only of unfounded speculation. This however would be a mistake. The calculation of imaginary quantities is of the greatest importance: questions frequently arise, of which we cannot immediately say, whether they include anything real and possible or not. Now, when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible.'

Much later, in para 700, the same point is reiterated, when a problem is attempted which leads to a quadratic equation with complex roots. Euler's next point is to show how the discriminant of a quadratic can be used to test for complex roots, so that the roots do not actually have to be determined.

It is in this section that Euler mentions irrational roots. He says that these can be found approximately but that complex ones cannot. He does not consider roots that are both complex and irrational.

In para 703 Euler makes a firm statement that quadratic equations have two roots, having mentioned in para 700 that the roots may be imaginary or impossible, and no value may be assignable to the unknown. He is thinking of impossible numbers as proper numbers, to be counted among roots together with reals, in spite of his feeling that they are non-existent. He does not give a general statement about the number of roots in a polynomial, but does give some examples of cubics. He solves the cubic $x^3 - 8 = 0$, obtaining as roots 2, and $-1 \pm \surd(-3)$ and verifies that these, when cubed give 8.

Euler says in para 703 (pp. 391, 393, 396) :

> 'It is true that these values are imaginary or impossible; but yet they deserve attention . . . every cube root has three different values; but that only one is real, or possible, the two others being impossible. This is the more remarkable, since every square root has two values, and since . . . a biquadratic has four different values, that a fifth root has five values and so on. In ordinary calculations, indeed, we employ only the first of those values because the other two are imaginary . . . . there is no doubt but that such an equation [the general cubic] has three roots after it has been seen . . . that this is true with regard to pure equations [of the type $x^n - a^n = 0$] of the same degree.'

These extracts show the state of Euler's thinking about complex numbers. The algebra of polynomials forced the conclusion that they are numbers, but he could find no logical place for them in the number system as he saw it. They cannot be approximated as can irrationals, and are founded upon negatives about which he was also somewhat confused. The claim for notice of this 'new species' lay in its admitted usefulness in allowing the mathematical treatment of problems that have no possible solution. None of Wallis's ideas are mentioned; had he known of it, he would surely have found the diagram helpful. He does not seem to have been thinking visually and may have thought a diagram neither possible nor desirable. He does not observe a narrow restriction of complex numbers to unsolvable problems; elsewhere he uses them freely in far more sophisticated ways, for the logarithms of negative numbers, for instance. His lack of confidence about their nature fortunately did not hinder him from making numerous extremely important discoveries about their behaviour, particularly their interactions with many branches of mathematics.

Euler's ideas about the nature of complex numbers show no advance beyond those of Wallis, on the contrary. In Euler's time it would be assumed, without the recognition that it was an assumption, that complex numbers can be ordered and described as positive or negative, as can reals. Some inkling of the fact that this is not so can be seen in the working of Descartes' rule of signs. The roots are distinguished as positive or negative with respect to the real part only and not with respect to the whole, and the rule can give contradictory information in the presence of different factors. Euler was familiar with the two-valued square root function $\sqrt{a}$ (a positive), where the two values are different and easily distinguishable. He would then have assumed that

$\sqrt{-1}$ was of the same kind. However, because of the automorphism $i \to -i$ in the complex number field, there is no relation which will distinguish between them. A distinction between i and -i can only be made if the restriction that it is the same i in each case is imposed. To Euler i would have had only one value, although it could have been positive or negative, and in the discussion of the errors in his Algebra, this is the assumption that has been made.

These errors had far-reaching consequences, particularly in England, and constitute a most important aspect of Euler's Algebra. It has been suggested that, as Euler was totally blind when the book was written, it may be that the mistakes were not his own, but those of his secretary or of his publisher. However this is not likely as the same error is repeated several times in different guises, and the erroneous idea carefully described. Slips are rarely found in Euler's work, but this error is particularly difficult to detect and he could not have done any proof-reading. It is most difficult to be sure that they do not originate with the 'apprentice', this depends on the extent to which Euler kept tight personal control over what was written in his name. The name of the apprentice was not given but it was evidently not Fuss (also recommended by Bernoulli) who did not arrive in St. Petersburg, where the book was written, until 1773[1]. In view of the consistency with which the errors occur, I think that it is likely that they are Euler's own. They were not corrected and caused much wavering of confidence in complex numbers among later mathematicians, some of whom were under the impression that the arithmetic of complex numbers was either ambiguous or not yet agreed upon. The accuracy of Euler does not seem to have been questioned.

Euler says in paras 148 and 149 of the Algebra (p.67) :

> 'Moreover as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{(-2)}$ multiplied by $\sqrt{(-3)}$ and $\sqrt{4}$ or 2 for the product of $\sqrt{(-1)}$ by $\sqrt{(-4)}$. We see, therefore, that two imaginary numbers, multiplied together, produce a real, or possible one.
> . . . it is evident . . . that $\sqrt{(+3)}$ divided by $\sqrt{(-3)}$ will give $\sqrt{(-1)}$ and that 1 divided by $\sqrt{(-1)}$ gives $\sqrt{\frac{+1}{-1}}$ or $\sqrt{(-1)}$.'

With the proviso mentioned above, $\sqrt{(-2)} \times \sqrt{(-3)}$ should be $-\sqrt{6}$, $\sqrt{(-1)} \times \sqrt{(-4)}$ is -2 , $\sqrt{(+3)} \div \sqrt{(-3)}$ is $-\sqrt{(-1)}$ and $\sqrt{\frac{+1}{-1}}$ is $-\sqrt{(-1)}$.

(1) Poggendorff, I, pp.821-23, (p.821)

In every case of multiplication and division Euler's text makes the same assumption. This is that $\sqrt{a} \times \sqrt{b} = \sqrt{(a \times b)}$ and $\sqrt{a} \div \sqrt{b} = \sqrt{(a \div b)}$ in every case. These assumptions are true in every case except the one where a and b are both negative in multiplication. Subject to the proviso described above, the result should be $i\sqrt{a}$ times $i\sqrt{b}$ giving $-\sqrt{ab}$, and not $+\sqrt{ab}$. When dividing, it is possible to obtain the correct result because the errors eliminate each other. Taking Euler's example, $\sqrt{(-4)} \div \sqrt{(-1)} = 2i \div i = 2$, also $\sqrt{\frac{-4}{-1}} = 2$ so his answer is correct. We know that Euler was obtaining his answers by the second method as this is how it is described when he finds $1 \div \sqrt{(-1)}$.

Present-day usage is that the symbol $\sqrt{\ }$ indicates the positive square root so Euler's $\sqrt{6}$ means today $+\sqrt{6}$, without ambiguity. His use of $\pm\sqrt{\ }$ in paras 700 and 703 shows that his practice was the same in these cases, although each has a preceeding number. Whatever Euler's usage, it is not possible to omit the '-' in $-\sqrt{\ }$ if the negative value is meant, there seems to be no alternative to attributing the error to Euler himself.

Some of the ideas current at this time first appeared in the correspondences between Euler, Goldbach, Daniel Bernoulli and Nicholas Bernoulli[(1)]. Goldbach and Euler corresponded on number theory in 1742. Functions having $\sqrt{-1}$ as an index such as $(2^{p\sqrt{-1}} + 2^{-p\sqrt{-1}})/2$ and series for expressions involving trigonometric functions were also discussed. Complex roots of quartics are mentioned several times and, in 1752, functions involving $\sqrt{-1}$ which generate reals, such as the sine and cosine. Daniel Bernoulli wrote to Goldbach in 1730, and to Euler in 1745, about integration by substitutions using $\sqrt{-1}$, but where the functions being integrated and the integrals were both real[(2)]. In 1731 Daniel Bernoulli and Euler corresponded on the complex formula for the area of a circle sector obtained by integration. In 1742 Euler was in correspondence with Nicholas Bernoulli and Goldbach on complex expressions for trigonometric functions such as, for sin B,

(1) Fuss, Correspondance, I, pp. 112, 113, 124-26, 133, 170, 201

(2) Fuss, II, pp. 376, 591

(3) Fuss, II, pp. 683, 687

'sinus arcus B = $(n - n^{-1})/2\sqrt{-1}$'; and series for trigonometric functions. Nicholas Bernoulli discussed the roots of the quartic $x^4 - 4x^3 + 2x^2 + 4x + 4$ in 1742 and 1743, giving the four complex roots $1 \pm \sqrt{2 \pm \sqrt{-3}}$ (1). Some of these ideas were included in the Algebra. There is nothing in these correspondences about the nature of complex numbers, they are treated only as entities with rules of behaviour that are being explored.

In 1749 Euler published his resolution of the controversy between Jean Bernoulli and Leibniz about the logarithms of negative and imaginary numbers (2). Bernoulli, supported by d'Alembert, held that $\log(-x) = \log(x)$, whereas Leibniz, with whom Euler agreed, gave arguments showing that this could not be so. Bernoulli advanced several fallacious arguments to show that $\log(-x) = \log(x)$, that is that $\log(-1) = 0$, one of which was that they must be equal as their differentials are equal. D'Alembert subscribed to the same error and de Missery corrected him with some forcefulness (see below). Leibniz objected that differentiation of logarithms only applied to positives and Euler pointed out that it would be disastrous if differentiation were not universal. Euler's argument was that the equality of the differentials meant only that the functions differed by a constant and not that they were equal. As $\log(-x) \neq \log(x)$, $\log(-1) \neq 0$, and Euler went on to show that $\log(\sqrt{-1}) \neq 0$ using a result discovered in 1702 by Bernoulli himself. By integrating $dx/(1+x^2)$ in two ways, Bernoulli had shown that $\frac{\sqrt{-1}}{2}\log\frac{(\sqrt{-1} - x)}{(\sqrt{-1} + x)} = \tan^{-1}x$, from which $\log\sqrt{-1} = \frac{\Pi\sqrt{-1}}{2}$. So $\frac{\log(\sqrt{-1})}{\sqrt{-1}}$ is the ratio of a quarter of the circumference of a circle to its radius and $\log\sqrt{-1}$ cannot be zero. One of Leibniz' arguments in favour of an imaginary value for $\log(-1)$ being imaginary depended upon the series $\log(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$ with $x = -2$, and Euler counters this with some examples showing that arguments depending on the properties of divergent series are unreliable. Although Euler does not support Leibniz' arguments, he agrees with his conclusion. Euler

(1) Fuss, II, pp. 691, 702

(2) Kline, p.409;
Euler, Opera Omnia, Ser.I, Vol.XVII, pp.195-232; (not seen);
Tahta, Imaginary Logarithms, pp.4-18

produced proofs that all logarithms are multi-valued, which removed the apparent contradictions. His first (unpublished) proof used Cotes' formula which can be seen to be periodic.

In his work on logarithms Euler manipulates $\sqrt{-1}$ confidently, according to its rules of behaviour. He used complex functions in other ways such as the evaluation of real integrals by separation of real and imaginary parts. He improved d'Alembert's demonstration that complex numbers are closed under exponentiation, and called this the fundamental theorem of complex numbers. Euler was able to advance complex number theory in many directions although it can be seen in his *Algebra* that he was fundamentally unsure about their nature.

Chapter IV

The Algebra (1769) of John Wallis to the Argand diagram (1806)

During this period the links between trigonometric, logarithmic and exponential functions were further explored and consolidated. Analysis gathered momentum, and problems in pure mathematics, physics and mechanics were being solved by means of the calculus with marked success. Complex function theory was being developed, Gauss gave proofs of the fundamental theorem of algebra and Wessel and Argand produced their diagrammatic representations for complex numbers. Mathematicians such as de Missery were confidently demonstrating the usefulness of complex numbers, while in England doubt and confusion about these and about negatives are discernible in the writings of Frend, Hutton and Playfair. It is easy to justify the claim that continental mathematicians were forging ahead more rapidly than British ones, particularly as far as complex numbers are concerned.

A French mathematical history written during this period was the General History of Mathematics of Charles (John) Bossut, which was published in French in 1802 and in English translation in 1803. It is a wordy account, almost totally devoid of mathematical symbolism. Bossut, who had the same teacher as Montucla and was also a pupil of d'Alembert, writes broadly on astronomy, optics, accoustics etc and their mathematical treatment, but with little detail. He mentions [1] the irreducible case of the cubic as having been paradoxical until Bombelli resolved the problem geometrically and compares Bombelli's demonstration with Plato's mean proportional method for the duplication of the cube. The imaginary parts of conjugates are described as destroying each other. Of arithmetic and algebra, Bossut says in Chapter I [2] :

> '. . . they are fundamentally one and the same science. Arithmetic operates immediately on numbers, and algebra operates in a similar manner on magnitudes in general.'

(1) Bossut, History, p.208

(2) Bossut, p.206

As there is virtually no mathematical symbolism in the book, it is difficult to tell whether Bossut is adhering to the Greek idea of the unknown as a magnitude, to the extent of excluding powers higher than three from algebra. Although he says that arithmetic and algebra are fundamentally the same, that is they have the same rules, he seems to be making the distinction that algebra does not apply to number. If this is not just an accident of expression or translation, this is a primitive attitude to find in a book of this date.

Bossut's attitude to complex numbers is similar to that of Montucla. Neither writer gave them much space in their histories, and neither makes any philosophical observations about their nature or their place in mathematics. The information given is elementary, more recent developments are ignored.

Edward Waring 1736-1798

Edward Waring was a mathematician working in England during this period. He became sixth Lucasian professor of mathematics at Cambridge while still in his twenties, and wrote a number of works on algebra. His work seems old-fashioned partly because, although written in the second half of the 18th Century, it was in Latin. Waring followed the ideas, notation and methods of Newton at a time when continental mathematicians were making great progress in the calculus along lines started by Leibniz. Lagrange, Euler and d'Alembert thought highly of him, but Hutton described some of his work as 'abstruse'[1]. Waring's writings have been considered to be poorly presented, confused, difficult to follow and full of typographical errors. He was a shy and modest man of high integrity, but lacked orderliness of thought. He suffered from severe myopia[1].

Waring's Meditationes Algebraicae was published in 1770. This is a detailed text-book of algebra at a non-trivial level, containing a number of interesting innovations. After some standard material about complex roots of a quadratic, the number of roots in a polynomial and Descartes' rule of signs, Waring gives the rule that if substitution of two values in a polynomial give positive and negative totals, then between them must lie a value giving zero (a root). In Chapter I he gives a method for finding greatest and least roots of a polynomial using multiplication by the terms of a series, which he says operates whether the roots are real or complex. In Chapter II he uses his series method to find the limiting numbers of positive, negative and complex roots, the limiting values between these roots and new equations with these limiting values as roots, from which Newton's and other rules can be deduced. Waring finds the number of complex roots in a polynomial by multiplying it by another having only real roots, and finds whether complex roots are positive or negative by multiplying the polynomial by $x + a$ and $x - a$. This a development of the property noted by Newton, that multiplication by another factor can change the

(1) Hutton, Dictionary (1815), II, p.584 ("Waring, Edward")

signs of complex roots. He gives a general rule for finding the number of complex roots using successive quadruples of terms instead of successive triples, as in Newton' rule. Waring then shows how to find the number of complex roots in an equation whose roots bear an algebraic relation to those of a given equation, such as where they are the squares or the squares of differences etc. He also deals with the number of complex roots in an equation in two or more unknowns. In the last two cases, Waring claims to obtain the exact number of roots, and not a limiting value for the number. He solves a cubic using de Moivre's theorem and says that Cardano's method involves three cubics, that is a resolution of nine dimensions. He shows how to rationalise irrational unknowns in an equation and considers one with imaginary coefficients. He multiplies together two quadratics with some imaginary coefficients to obtain a quartic with real coefficients and refactorises this into quadratics with real coefficients. This is to demonstrate that every algebraic equation with real coefficients can be factorised into quadratic and linear factors with real coefficients. Waring then gives an iterative method for improving approximations for roots, whether real or complex. If the approximate root is $a + b\sqrt{-1}$, substitute $x = a + a' + (b + b')\sqrt{-1}$, and reject higher powers of a' and b'. Equating real and imaginary parts gives values for a' and b'.

Waring considers only the algebra of polynomials in this book. He does not use geometrical methods, but demonstrates the power of the arithmetic series approach. He gives no discussion about the nature of complex numbers, but uses them to great effect. He gives no graphical representations of any sort. His use of words is confusing, both 'irrationalibus' and 'impossibilibus' are used to mean imaginary. This Algebra represents a real advance in the algebra of polynomials and it is unfortunate that it was not easier to read.

In the Meditationes Analyticae of 1773, Waring covers Newtonian fluxions, giving some rules for differentiation and a number of integrals. Some of the examples include complex numbers, for instance he gives the integral of $\dfrac{a^2\dot{x}}{a^2 + x^2}$ as the complex quantity $\dfrac{\sqrt{-a^2}}{2}\log \dfrac{x + \sqrt{-a^2}}{x - \sqrt{-a^2}}$.

He makes use of complex numbers several times, referring to them as 'imaginaria quantitas', but gives no discussion of their nature or his views of them.

Waring's most important contributions to mathematics were in number theory in which he made several useful conjectures and deductions, and in the treatment of sequences, as he was one of the first mathematicians to recognise that the convergence of these needs consideration when manipulating or summing them. He uses series with polynomials to great effect. He took a broader view of coefficients and unknowns in polynomials than many mathematicians and his use of complex numbers shows that he was able to accept them as numbers. He does not deal with problems in his Algebra, but builds up the subject as a discipline in itself, which did not require 'usefulness' as a justification. This represents a much more advanced view of algebra, and of complex numbers, than was exhibited by many mathematicians of his time.

William Frend 1757-1841 , Baron Francis Masères

William Frend, whose daughter Sophia Elizabeth married Augustus de Morgan, was outspoken in condemnation of the use of negatives in algebra. Frend's reading of the Bible had led him to believe that he had been hoodwinked by the Church into an acceptance of the Trinity, and this discovery led him to question other beliefs, particularly in mathematics(1). He came to think of algebra containing negatives as an art rather than a science, and advocated the elimination of negatives from algebra in order to restore to it the status of a science. Similar views were expressed in similar language by de Morgan, and the two must have discussed these matters. The line taken by Frend was that there was no proper definition of negatives, however this argument cannot be regarded as carrying any weight as no part of the number system had been defined at the time.

Frend's algebra text-book Principles of Algebra was published in 1796 and he sets out his philosophy of numbers in the preface. His views are very decided, '-' means a subtraction and can only be applied when the result would not be negative. His extraordinary aim was to write an algebra of non-negative numbers. Frend uses the word 'impossible' for complex and imaginary numbers, but of course, under his system they can never arise. From the preface (2) :

> '. . . to attempt to take [a number] away from a number less than itself is ridiculous.
> Now when a person cannot explain the principles of a science without reference to metaphor, the probability is, that he has never thought accurately upon the subject.
> . . . algebraists, who talk of a number less than nothing, of multiplying a negative number into a negative number and thus producing a positive number, of a number being imaginary . . . they talk of two roots to every equation of the second order, and the learner is to try which will succeed in a given equation: they talk of solving an equation, which requires two impossible roots to make it solvible: they can find out some impossible numbers, which, being multiplied together, produce unity. This is all jargon, at which common sense recoils; but from its having been once adopted, like many other figments, it finds

(1) Pycior , Historia Mathematica, 9(1982),393

(2) Frend , Algebra, preface, p.x

> the most strenuous supporters among those who love to take things upon trust, and hate the labour of serious thought. [Complex answers are attributable to] either an error in the mode of reasoning, or to false premisses.'

The 'metaphors' complained of are those of book-debts, receding tides etc in Maclaurin's Algebra, described above. These remarks are quite uncompromising, at least three centuries of progress in the number system are rejected and the writing of an algebra book under the restrictions described seems a considerable achievement.

Frend gives methods for solving simple equations, but his determination to eliminate negative numbers leads to cumbersomeness and much multiplicity of methods. Negatives do not arise either in equations or as roots, and they are also avoided in the calculations.

Frend takes the cubic $x^3 - qx + r = 0$, and shows how to solve it by letting $x = a + b$ with $3ab = q$. This gives $a^3 + b^3 + r = 0$ leading to $a^6 + q^3/27 + ra^3 = 0$, a quadratic in a^3. The solution is

$$x = \sqrt[3]{\sqrt[2]{\frac{r^2}{4} - \frac{q^3}{27}} - \frac{r}{2}} + \sqrt[3]{-\frac{r}{2} - \sqrt[2]{\frac{r^2}{4} - \frac{q^3}{27}}}$$

Frend says (1) :

> '1. Let the equation be $x^3 + 27x - 28 = 0$, in which x is equal to one, and consequently a and b must each be less than one, and 3ab cannot be equal to 27. Hence it is evident that this method cannot be applied to a vast variety of equations, in which the unknown number is incapable of being divided into two parts, so that three times their product should be equal to q.
>
> 2. After having made the supposition that $3ab = q$, an equation is formed $a^3 + b^3 + r = 0$. Now this is absurd; for three numbers added together cannot be equal to nothing.
>
> 3. From absurd premisses, an absurd conclusion is most likely to follow, and this is seen in the expression $b = \sqrt[3]{-\frac{r}{2} - \sqrt[2]{\frac{r^2}{4} - \frac{q^3}{27}}}$, an expression which has no meaning.'

The three numbers in para 2 cannot sum to zero because none can be negtive; in para 3, if r and the square root in b are both positive, then the quantity whose cube root is required and the root itself are both negative, and so inadmissable.

(1) Frend, p. 211

Frend's Algebra has a comprehensive appendix by Baron Francis Masères, who also contributed to Hutton's Dictionary. Masères was considered for the sixth Lucasian professorship of mathematics at Cambridge in competition with Waring, but was not successful [(1)]. Masères takes every case of the cubic, giving for each a solution which does not introduce negatives, with an example. This circumvents Frend's difficulties with the cubic. For the irreducible case Masères recommends the Newton-Raphson method, in which an approximate root is required as a first estimate and improved iteratively [(2)]. Considering $bx - x^3 = c$, using a small increment in x, he shows that $c \leq \frac{2b\sqrt{b}}{3\sqrt{3}}$ and if $c = \frac{2b\sqrt{b}}{3\sqrt{3}}$ or $\frac{cc}{4} = \frac{b^3}{27}$, the equation will have one root $\frac{\sqrt{b}}{\sqrt{3}}$, if c is less then there will be two roots α and β, where $\alpha < \frac{\sqrt{b}}{\sqrt{3}}$ and $\frac{\sqrt{b}}{\sqrt{3}} < \beta < \sqrt{b}$. He takes a first estimate close to one of these limits and uses the Newton-Raphson method to improve it, giving a great many examples. In Part II of the book, Frend describes the rule of double false position for finding roots. In this method the first estimate does not need to be so close to the root.

The book is necessarily long and tedious, it goes against the usual trend in mathematics towards generalisation and simplification of procedures. No rigorous demonstration of the behaviour of negatives under, for instance, multiplication was available at the time and one must have a certain sympathy for Frend's attitude. He has been described as eccentric, but there is a sense in which he was right. Frend has overlooked the fact that no rigorous demonstrations had been given for positives either, if negatives need rigour so do all numbers. The methods used to avoid negatives are ingenious if lengthy, but the elimination of these neatly solves the problem of how to deal with imaginaries. Frend lived for many years after the publication of the Algebra ; I have not been able to establish whether he ever changed his views about negatives in algebra.

(1) Hutton (1815), II, p. 584

(2) Frend, p. 292

Charles Hutton 1737-1823 , John Playfair 1748-1819

The far-reaching consequences of the 'errors' in Euler's <u>Algebra</u> referred to above are well exemplified in some of the writings of Charles Hutton. Hutton was a competent mathematician who rose from humble beginnings as the self-educated son of a Northumberland colliery worker, to become Professor of Mathematics at the Royal Military Academy Woolwich, a position which he won in open competition. On retirement from this post, he was awarded a pension of £500 p.a. by the Board of Ordnance, an indication of how highly his services had been valued. In 1798 he published <u>A course of Mathematics composed and more especially designed for the use of the gentleman cadets</u> , in two volumes. Each volume consists of three parts. Volume I has sections entitled 'Arithmetic', 'Logarithms and Algebra' and 'Geometry'; volume II has sections entitled 'Trigonometry', 'Conics' and 'Mechanics'. There is a substantial treatment of the Newtonian 'Doctrine of Fluxions' in the mechanics section, in which the language and notation are those of Newton. This is an excellent, comprehensive and easily-followed exposition of the mathematics then available , and at a suitable level for military cadets. The arithmetic section gives computational methods for nth roots, but there is no mention of the even roots of negative numbers, either at any point in the text, or in the introduction. If there were no other evidence, we should have to speculate about this omission. Hutton could not have been unaware of the existence of complex numbers and the level of the work, and its breadth, were such that a treatment of complex quantities would have made a very satisfactory completion to the picture given of mathematics. But the aim of the book was neither to give a picture of mathematics nor a comprehensive mathematical education, but to equip cadets with an adequate knowledge of mathematics for military needs. It might be supposed that Hutton had considered this point and decided that a knowledge of complex numbers was unlikely to be useful to an army officer.

However in 1806 a third volume was produced by Hutton and 'Dr Gregory', also of the Royal Military Academy. The new chapters are to be interposed between those in Volumes I and II. Chapter VIII is entitled 'On the nature and solution of equations in general', and covers methods of solving quadratics and cubics including trigonometrical methods, with acknowledgement of Cardano and Euler. From article 5 (1) :

> 'It sometimes happens that an equation contains imaginary roots . . . This class of roots always enters an equation by pairs: because they may be considered as containing, in their expression at least, one even radical before a negative quantity, and because an even radical is necessarily preceded by the double sign $\pm$.'

There follow remarks about numbers of roots, conjugates, determination of roots etc. There is no mention of uncertainties in the arithmetic of complex numbers, and nothing about their nature. The word 'real' is used in the modern sense. There is no indication as to which parts of Volume III were written by Hutton and which by Gregory, but in view of Hutton's doubts about complex numbers as revealed in his Dictionary , this chapter was probably contributed by Gregory.

A somewhat unfortunate post-script to this book appears in Daniel Dowling's Key to Hutton's Course of Mathematics of 1818. The Key consists of solutions to problems in the Course and Dowling's only observations concern the frequency with which his answers differ from those of Hutton.

Hutton's Mathematical and Philosophical Dictionary of 1796 gives a clear picture of his views on negative and complex numbers, and on mathematics generally. The format of the Dictionary is two columns to a page, which is of approximately A4 size. Some of the entries are as follows, more general entries being given for comparison :

Complex	No entry
Impossible	'. . . same as Imaginary . . . which see'
Negative	$1\frac{1}{2}$ columns
Root	3 columns
Imaginary	$3\frac{1}{2}$ columns
Equation	11 columns
Algebra	68 columns
Zero	No entry
Integers	4 lines
Irrational	'. . . see surds'
Limit	$\frac{1}{3}$ column
Variable	$\frac{1}{3}$ column
Euclid	1 column
Geometrical, Geometry	4,3 columns

(1) Hutton, Course of Mathematics, III, p. 175

Surd	$3\frac{3}{4}$ columns
Number	4 columns
Triangle	5 columns
Trigonometry	14 columns

Hutton's difficulties started with negative numbers; he shared Frend's scepticism about these to some extent. He says under 'Negative' (1):

> 'The use of the negative sign in algebra; is attended with several consequences that at first sight are admitted with some difficulty and has sometimes given occasion to notions that seem to have no real foundation. . . . The theorems that are sometimes briefly discovered by the use of this symbol may be demonstrated without it by the inverse operation, or some other way; and though such symbols are of some use in the computations in the method of fluxions etc. its evidence cannot be said to depend upon any arts of this kind.'

The fact that results can be proved another way does not give Hutton confidence in the reliability of negatives any more than than it gave other mathematicians confidence in complex numbers.

Under 'Root' Hutton describes square, cube roots etc. as mean proportionals between one and the given number. In a subsection on 'Real and imaginary roots' he describes how 'imaginary or impossible' roots arise as the even roots of negative quantities.

It is in the entry under 'Imaginary' that Hutton gives what must have been the true reason for omitting complex numbers from the Course . Here he lays out the various versions of complex arithmetic of which he knows. It does not seem to have occurred to him that any of the mathematicians mentioned might have made a slip. Perhaps his doubts about negatives prepared him to expect ambiguities in complex number arithmetic. Under 'Imaginary' (pp.625-26) :

> 'The arithmetic of these imaginary quantities has not yet been generally agreed upon; viz as to the operations of multiplication, division and involution; some authors give results with + , others on the contrary with the negative sign − . Thus Euler in his Algebra . . . makes the product of two impossibles when they are unequal to be possible and real as $\sqrt{(-2)} \times \sqrt{(-3)} = \sqrt{6}$ and $\sqrt{(-1)} \times \sqrt{(-4)} = \sqrt{4}$ or 2. But how can the equality or inequality of the factors cause any difference to the signs of the products ? If $\sqrt{(-2)} \times \sqrt{(-3)}$ be $= \sqrt{(+6)}$ how can $\sqrt{(-3)} \times \sqrt{(-3)}$ . . . be -3 ? . . . Also in division he makes $\sqrt{(-4)} \div \sqrt{(-1)}$ to be $= \sqrt{(+4)}$ or 2 and $\sqrt{(+3)} \div \sqrt{(-3)} = \sqrt{(-1)}$ also that 1 or $\sqrt{(+1)} \div \sqrt{(-1)} = \sqrt{\frac{+1}{-1}} = \sqrt{(-1)}$ consequently multiplying the quotient root $\sqrt{(-1)}$ by the divisor $\sqrt{(-1)}$, must give the dividend $\sqrt{(+1)}$ and yet by squaring he makes the square of $\sqrt{(-1)}$ or the product $\sqrt{(-1)} \times \sqrt{(-1)} = -1$.
>
> But Emerson makes the product of imaginaries to be imaginary; and for this reason, that "otherwise a real product would be

(1) Hutton , Dictionary (1796), II, p. 147

raised from impossible factors which is absurd. Thus
$\sqrt{(-a)} \times \sqrt{(-b)} = \sqrt{(-ab)}$ and $\sqrt{(-a)} \times \sqrt{(-b)} = -\sqrt{(-ab)}$ etc.
also $\sqrt{(-a)} \times \sqrt{(-a)} = -a$ and $\sqrt{(-a)} \times -\sqrt{(-a)} = +a$ etc."

And thus most of the writers on this part of Algebra, are pretty equally divided, some taking the product of impossibles real, and others imaginary.

. . . Mr Playfair. . . makes the product of $\sqrt{(-1)}$ by $\sqrt{(-1)}$ or the square of $\sqrt{(-1)}$ to be -1; and yet in another place he makes the product of $\sqrt{(-1)}$ and $\sqrt{(1 - z^2)}$ to be $\sqrt{(-1 + z^2)}$.(1) Mr Playfair concludes "that Imaginary expressions are never of use in investigations, but when the subject is a property common to the measures both of ratios and of angles; but they never lead to any consequence which might not be drawn from the affinity between those measures and that they are indeed no more than a particular method of tracing that affinity. . . . the arithmetic of impossible quantities will always remain an useful instrument in the discovery of truth and may be of service when a more rigid analysis can hardly be applied. . . . M. Bernoulli has found, for example, that if r be the radius of a circle, the circumference is $= \frac{4\log\sqrt{(-1)}r}{\sqrt{(-1)}}$. Considered as a quadrature of the circle, this imaginary theorem is wholly insignificant, and would deservedly pass for an abuse of calculation; at the same time learn from it, that if in any equation the quantity $\frac{\log\sqrt{(-1)}}{\sqrt{(-1)}}$ should occur, it may be made to disappear, by the substitution of a circular arch . . . The same is to be observed of the rules which have been invented for the transformation and reduction of impossible quantities; they facilitate the operations of this imaginary arithmetic; and thereby lead to the knowledge of the most beautiful and extensive analogy which the doctrine of quantity has yet exhibited. . . . The real and Imaginary roots of equations may be found from the method of fluxions applied to the doctrine of maxima and minima . . . but when the equation is above three dimensions, the computation is very laborious.'

The last remark refers to Newton's rule for the number of complex roots in an equation.

Summarising the methods Hutton has collected for multiplying imaginaries and including Bombelli's version, he has (in modern notation) :

$\sqrt{(-a)} \times \sqrt{(-b)}$	$= -\sqrt{(ab)}$	Bombelli
	$= \sqrt{(ab)}$ and $\sqrt{(-a)} \times \sqrt{(-a)} = -a$	Euler
	$= \sqrt{(-ab)}$ and $-\sqrt{(-ab)}$	Emerson

The product $\sqrt{(-a)} \times \sqrt{(-b)}$ has been given as real and positive, real and negative, imaginary and positive and imaginary and negative by various writers known to Hutton. He clearly considers that these versions show that the algebra of complex numbers 'has not yet been generally agreed

(1) Playfair, Works III, p. 8

upon'. This seems to imply that he thinks that the problem may be resolved by agreement rather than rigorous mathematical processes, and has an optional quality. It is difficult to believe that this was really his view, this impression may simply be the result of an unfortunate choice of words. It is necessary to decide on the convention in which $\sqrt{-2} = i\sqrt{2}$ and $\sqrt{-3} = i\sqrt{3}$ involves the same i, but this is the only optional element. Once chosen, the pattern of the algebra of $\sqrt{-1}$ is fixed. The automorphism in question would mean little to Hutton who would be thinking in terms of a single fixed value for $\sqrt{-1}$. The quoted remark by Emerson shows that he at least was unaware of the property of conjugates, that their product is real, a fact well-known from at least the time of Bombelli. Hutton's quotation of this remark shows that he too was open to doubt on this point. Emerson was evdently not a reliable algebraist and it is unfortunate that Hutton should take his opinion seriously enough to quote in the Dictionary.

The quoted remark of Playfair that Bernoulli's result is 'wholly insignificant', is also unfortunate, although he does say in relation to the quadrature of the circle. It shows Playfair's limitations as, implicit in it, is the relation $e^{i\Pi} + 1 = 0$, obtainable if the inverse nature of the exponential and logarithmic functions is understood.

John Playfair, whose main interest was geology, was Professor of Natural Philosophy at the University of Edinburgh(1) and he expresses somewhat philosophical views of negative and complex numbers. He sees the fact that algebra can deal with quantities that cannot be represented in geometry as a weakness rather than a strength in algebra. He finds it a paradox that results obtained using such algebra are borne out by geometry, so it is not as useless as it ought to be. He is undecided whether algebra is an art or a science. He says symbols cannot form part of a science nor manipulation of them part of an art. He finally finds algebra acceptable for the usual reason, that it is useful. Playfair says, in On The Arithmetic of Impossible Quantities(2):

(1) Playfair, Works, title page

(2) Playfair, pp. 1-8

'The paradoxes which have been introduced into algebra, and remain unknown in geometry, point out a very remarkable difference in the nature of those sciences. The propositions of geometry have never given rise to controversy(1), nor needed the support of metaphysical discussion. In algebra . . . the doctrine of negative quantities and its consequences have often perplexed the analyst . . . the geometer is never permitted to reason about the relations of things which do not exist, or cannot be exhibited. In algebra again every magnitude being denoted by an artificial symbol, to which it has no resemblance . . . the analyst continues to reason about the characters after nothing is left which they can possibly express: if then, in the end, the conclusions which hold only of the characters be transferred to the quantities themselves, obscurity and paradox must of necessity ensue. . . . they have been made the subjects of arithmetical operations . . . and, what may seem strange, just conclusions have in that way been deduced. . . . the arithmetic of mere characters can have no place in a science. . . . Is investigation an art so mechanical, that it may be conducted by certain manual operations ? Or is truth so easily discovered, that intelligence is not necessary to give success to our researches ?' (Trans., Roy. Soc., London 1779)

To Playfair, algebra must represent the arithmetic of real positive numbers to be valid. He cannot accept the move towards symbolism that has begun to take place. To him, complex and negative numbers are baffling because non-existent but he does not consider the existence or otherwise of, for instance, the naturals. The acceptable number categories are those that are geometrically constructable, and no doubt Playfair would use this property in demonstrating their existence.

Hutton's position is more advanced and open-minded than that of Playfair. Hutton's Dictionary shows the beginning of the transition in the use of the word 'impossible' from numbers to problems. At some points numbers are referred to as 'imaginary' and problems as 'impossible', but these usages are not yet fixed and there is still a certain amount of interchange.

Perhaps the most important factor in the inability of some mathematicians to resolve the confusion about complex numbers was the fact that a symbol for $\sqrt{-1}$ was not yet in general use, although 'i' had been used by Euler from the mid-18th Century. By writing $\sqrt{(-a)} \times \sqrt{(-b)}$ as $i\sqrt{a} \times i\sqrt{b}$, it is easier to see that the result should be $ii\sqrt{a}\sqrt{b}$ or $-\sqrt{ab}$. The associative and commutative laws would have been used intuitively.

(1) Open to question (DW)

Waring had made useful contributions in algebra but, because of their obscurity, these insights did not become widely known. The picture emerging in Britain at the end of the 18th Century is one of serious confusion in the minds of some mathematicians about complex numbers. It was most unfortunate that the reliable summary given by Bombelli was overtaken by errors made by Euler, usually also entirely reliable. This left the way clear for others to express opinions. Hutton does not say anywhere that writers he quotes may have been mistaken. We might not expect Euler to have been wrong, such was his reputation, but with hindsight we would want to examine closely any unsubstantiated remarks made by Emerson. Having identified a dilemma, Hutton does not seem able to make a decision between the supposed alternatives. His hint that it might be a matter of choice can easily be seen as lack of confidence in the structure of algebra. Euler's Algebra was widely read both on the Continent and in England, but it does not seem to have caused the problems in Europe that it did in England. It may be that on the Continent the errors were recognised as simple slips, or possibly, were just not noticed.

Joseph-Louis Lagrange 1736-1813 , Pierre-Simon Laplace 1749-1827

Lagrange and Laplace both gave lectures at the short-lived École Normale in Paris during 1795. The lectures in the mathematics faculty were of a high standard and provide a useful guide to the status of complex numbers in France at the end of the 18th Century. Neither mathematician made innovations in complex number theory, but both made bold use of them. As with many mathematicians, although they thought complex numbers baffling and to be avoided if possible, they both felt obliged to accept them on the basis of their usefulness.

Lagrange discusses complex numbers as roots of the cubic and the paradox of the irreducible case. To him 'number' is synonymous with 'real number' and the test of existence for a number is whether it can be constructed geometrically. The paradox of the irreducible case is that, as the complex expression represents a real number, it <u>can</u> be constructed geometrically which means that it is also valid in algebra. Unfortunately he does not discuss here the problem of representing negatives geometrically, his attitude to this would make an interesting comparison. Lagrange says of the irreducible case(1):

> 'But how is this value of x to be assigned ? It would seem that it can be represented only by an imaginary expression or by a series which is the development of an imaginary expression. Are we to regard this class of imaginary expressions, which correspond to real values as constituting a new species of algebraic expressions which although they are not, like other expressions susceptible of being numerically evaluated in the form in which they exist, yet possess the indisputable advantage - and this is the chief requisite - that they can be employed in the operations of algebra exactly as if they did not contain imaginary expressions [?] They further enjoy the advantage of having a wide range of usefulness in geometrical constructions as we shall see in the theory of angular sections so that they can always be exactly represented by lines; while as to their numerical value, we can always find it approximately to any degree of exactness that we desire . . . . We may regard it as a demonstrated truth that the general expression of the roots of an equation of the third degree in the irreducible case cannot be rendered independent of imaginary quantities.'

(1) Lagrange, <u>Lectures</u> , pp.54-95 (p.79)

Lagrange shares the general uncertainty as to whether the roots in the irreducible case are a new species of number. In a sense they are a species different from reals, that is complex, but this difference is more apparent than genuine as they reduce to reals. Lagrange does not take these properties as vindication of the case for complex numbers. Lagrange attempted to prove the fundamental theorem of algebra but, as he says later in this lecture, he could devise no proof that did not lead to a circular argument.

Laplace's attitude to complex numbers was similar to that of Lagrange. He defines imaginary numbers and describes the form taken by roots. He gives the properties of conjugates and of the roots in the irreducible case. He stresses the usefulness of complex numbers, especially the equating of real and imaginary parts in analysis. In the Quatrième Séance of the 1795 lectures[1] Laplace solves the equation $3x - x^2 = 2$, and obtains the roots $x = \frac{3}{2} \pm \frac{\sqrt{-3}}{2}$. He then talks about real and imaginary quantities [2] :

> 'La quantité $\sqrt{-3}$ est impossible; car un nombre réel, positif ou négatif, ne peut avoir pour carré un nombre négatif; le problème qui conduit à ces valeurs est donc impossible. Ces valeurs se nomment imaginaire; on peut les mettres sous la forme d'une quantité réelle, augmentée ou diminnée d'üne autre quantité réelle multipliée par $\sqrt{-1}$; . . .
>
> Quoique les quantités imaginaires soient impossibles, cependent leur considération est du plus grand usage dans l' Analyse. Souvent les grandeurs réelles se presentent sous la forme de plusiers imaginaires, dans lesquelles tout ce qu'il y a d'imaginaire se detruit mutuellement quoiqu'il soit difficile de le reconnaître à l'inspection des formules. On verra bientôt que l'expression des racines des équations du troisieme degré est dans ce cas, lorsque toutes les racines sont réelles; d'ailleurs, la comparison des grandeurs réelles entre elles, et des imaginaires avec les imaginaires, est un moyen fécond de l'Analyse, pour déterminer les grandeurs.'

(1) Laplace, Oeuvres , XIV, pp. 10-178

(2) Laplace, p. 45

To Laplace an imaginary consists of a real and an imaginary part, either of which may be positive or negative. He describes in this section the properties of conjugates. In the Cinquième Scéance he discusses the nature of roots, the form taken by complex roots and says that there can be no roots that are not either real or complex, the equivalent of d'Alembert's result[1]. In this session he acknowledges Waring's Meditationes Algebraicae and some of Gauss's ideas. The separation and equating of real and imaginary parts is covered in the Sixième Séance[2], and in the Huitième Séance[3] he gives some useful substitutions. The problem under discussion is the division of angles into equal parts, and Laplace uses de Moivre's theorem with the substitutions $\frac{1}{2}(\cos x + \sqrt{-1}\sin x) + \frac{1}{2}(\cos x - \sqrt{-1}\sin x)$ for cosx, and $\frac{1}{2\sqrt{-1}}(\cos x + \sqrt{-1}\sin x) - \frac{1}{2\sqrt{-1}}(\cos x - \sqrt{-1}\sin x)$ for sinx. De Moivre's theorem is also used to find the factors of $x^n - a^n$ and $x^n + a^n$.

Both Lagrange and Laplace were interested in the applications of mathematics, they saw it as a useful tool for solving difficult problems in mechanics and physics. Both saw the value of complex numbers as residing in their usefulness, and stressed this point in their lectures.

(1) Dhombres, Rev.Hist.Sci., 33(1980), 314-48 (p.336)

(2) Laplace, pp. 66-77 (p.76)

(3) Laplace, pp.101-132 (p.106)

Louis Arbogast 1759-1803

Louis Arbogast was professor of mathematics at Strasbourg, and his book Du Calcul des Derivations was published there in the year VIII, that is 1800. He makes frequent use of the symbol 'D' which he calls a 'signe des derivations', of which the modern '$\frac{d}{dx}$' is a particular case. Waring had previously used D to mean $\frac{d}{dx}$ (see above), although other writers had used it to represent a finite difference. On page xxj of the preface Arbogast gives a 'Tableau des notations principales' in which D, D^{-1}, D^{-n}, ∂, ∂^{-1}, ∂^{-n}, Δ, Σ, $\underset{c}{\partial}$, $\int$, d and $\underset{c}{D}$ are given, sometimes in conjunction with various prefixes and suffixes. The relationships $\int = d^{-1}$ and $\int^n = d^{-n}$ are also given. Arbogast says that he is generalising Lagrange's analysis, of which differential calculus methods are only a particular case, and claims his ideas as a great simplification. He has a simplified notation and says that the secret of the strength of analysis lies in the happy choice of 'signs' (that is, notation). He says in the preface that the rules for deriving the quantities which depend on the function are the same as those of the differential calculus for taking successive differentials of a function. The differential of the variable is constant and equal to unity, which means that it can always form part of these quantities. That is, since dx = 1, then $D(x)^2$ can be taken as 2x.dx or simply 2x. Later d^{-1}, d^{-2}, etc and D^{-1}, D^{-2} etc are described as meaning respectively 'differentiales inverses' and 'derivées inverses'. This seems to contradict the relationship $\int = d^{-1}$ etc, unless it can be taken that $\int d = 1$, with no arbitrary constant.

Arbogast emphasises what he describes as the simplicity of his methods acknowledging many previous writings, such as the Meditationes Analyticae of Waring and his series methods, and the method of exhaustion of the 'Ancients'. In a footnote, referring to his paper of 1789, he shows how to obtain any function of x as a power series in Δx, following Lagrange's method. Arbogast's book is divided into six Articles. In the first he sets up a series assuming that $F(\alpha + x) = a + \frac{bx}{1} + \frac{cx^2}{1.2}$ + etc finding a, b, c etc by differentiating and setting x = 0.

Articles two to five are about expanding polynomials in series, and article three shows that the product of two series is another series. Article six is about differentiation and begins to show the point of expanding functions in series when they are to undergo further manipulations such as integration.

Arbogast is meticulous in acknowledging the work of other mathematicians and, among others, mentions Leibniz, Waring, Lagrange and Laplace. This book shows his great skill, not only in manipulating series, but in devising and operating with new notations. His use of D, d etc is a very early example of the separation of operator and operand in the differential calculus. This method was new, and proved very fruitful when developed in the early 19th Century. Arbogast stresses repeatedly the simplicity of his methods, but it would be fair to say that this simplicity was bought at the expense of great proliferation of symbols. In this book he lays down a simple system for real analysis, so unfortunately complex numbers are not brought into the discussion at any point. This is a pity as Arbogast was an adventurous innovator of symbolism and an interesting operational treatment of complex numbers might have been hoped for. Further research might prove rewarding.

Antoine Suremain de Missery 1767-1840

An illuminating summary of the state of the algebra of complex numbers in France at the end of this period is given by Antoine Suremain de Missery in Théorie Purement Algébrique des Quantités Imaginaires et des fonctions qui en résultent, published in Paris in 1801. The author is described on the title page as 'ci-devant Officier d'Artillerie, de la Société des Sciences de Paris et de celle de Dijon'.

In this work de Missery starts by claiming that he will use only simple algebra, that is, he will not be using either geometry or infinitesimals (calculus), with the implication that algebra is easier and superior. In this he is taking a similar line to Arbogast, who was also trying to simplify the analysis of Lagrange. De Missery discusses the controversy between Leibniz and 'l'un des Bernoulli'[(1)] (Jean 1667-1748), on the nature of the logarithms of negative quantities which the former, supported by Euler, considered imaginary and the latter, supported by d'Alembert, thought real. He describes the uncertainty of 'vulgar' mathematicians and says that he agrees with Euler who takes the view that positive reals each have an infinite number of logarithms all imaginary except one, and negative reals an infinite number all imaginary. He proposes to raise some extraordinary mistakes of d'Alembert, whose hypotheses that $\log(-x) = \log x$ and $\log(-1) = 0$, are inadmissible. De Missery seems to be taking pleasure in pointing out d'Alembert's errors. He then talks about imaginary exponents and the functions that might result. These functions, when applied to the arc of a circle of radius one, give the sine, cosine, tangent, cotangent, secant and cosecant of the arc and other properties of circles, spheres and triangles both rectilinear and spherical, and formulae in both trigonometries very 'elegantly'. He says he will obtain the logarithm of an imaginary quantity such as $A + \sqrt{-1}B$ using only ordinary algebra, whereas d'Alembert and Euler had used calculus and geometry, and Foncenex had used algebra and geometry. D'Alembert thought that algebra alone

(1) De Missery, Théorie, pp.1-3

would be insufficient without geometry, but de Missery's method involves expressing arcs of circles in terms of imaginary logarithms (Jean Bernoulli's result, quoted by Hutton in his Dictionary from Playfair, see above). The use made of this result by de Missery is in marked contrast to the dismissive remarks of Playfair. The 'ordinary' algebra required includes some series and their inverses. These are (p. 12) :

$$\log(x + s) = \log x + \frac{s}{x} - \frac{1s^2}{2x^2} + \frac{1s^3}{3x^3} - \frac{1s^4}{4x^4} + \quad \text{etc}$$

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{2.3} + \frac{z^4}{2.3.4} + \quad \text{etc, } e \text{ being the base,}$$

or, more generally

$$\log(x + s) = \log x + A\left(\frac{s}{x} - \frac{1s^2}{2x^2} + \frac{1s^3}{3x^3} - \text{etc} \right) \quad A \text{ being the modulus}$$

$$c^z = 1 + \frac{z}{A} + \frac{z^2}{2A^2} + \frac{z^3}{2.3.A^3} + \quad \text{etc} \qquad c \text{ being the base}$$

After expansion of $e^{y \pm z\sqrt{-1}}$, de Missery shows that if $\log x = y$, then $\log(-x) = y \pm \Pi\sqrt{-1}$ belongs to the same system as $e^y = x$, so y is not now the logarithm of the two different quantities log x and log(-x) (pp.23-25). Similarly if log(-x) is taken as y, then $\log x = y \pm \Pi\sqrt{-1}$. An objection is anticipated, for

$\log(a) = \log(\text{real } a)$ where 'real' means positive

$\log(-a) = \log(\text{real } a) + \log(-1)$ taking $-a = a \times (-1)$, a positive

$\log(a) = \log(\text{real } a) + \log(-1) + \log(-1)$ taking $-a = (-a) \times (-1)$

$= \log(\text{real } a) + 2\log(-1)$ from which can be obtained the result

$2\log(a) = 2\log(-a)$ so $\log(a) = \log(-a)$.

However this does not indicate that log(a) = log(-a), but only that the sum of two particular values of log(a) is the double of log(-a). The same result is obtained starting with $\log(-a)^2 = \log(\text{real } a)^2$. So de Missery holds with Leibniz and Euler against Bernoulli and d'Alembert, that the logarithm of a negative is imaginary, and that $\log(x) \neq \log(-x)$. He notes also that it is possible to have, in an infinite number of ways (p. 27), the sum of

2 different logs of $-a$ = that of two different logs of a

or 2 " $-a$ = the double of log a

or 2 " a = " a

or 2 " a = " $-a$.

He further labours the point about the error in d'Alembert's calculus method as follows (p. 28) :

d'Alembert takes $d(\log x) = \frac{dx}{x}$ and $d(\log(-x)) = \frac{-dx}{-x} = \frac{dx}{x}$

so that $d(\log x) = d(\log(-x))$ and $\log x = \log(-x)$ by integration. 'But he knows better than I do that the complete integral is $\log(-x) = \log x$ + constant' and the constant is $\log(-1)$, which d'Alembert knows is $\pm \text{II}\sqrt{-1}$.

In the next section (pp. 33-46) de Missery shows that :

$$\log(x + s\sqrt{-1}) = \tfrac{1}{2}\log(x^2 + s^2) + (q \pm 2k\text{II})\sqrt{-1} \quad \text{where } x > 0$$
$$= \tfrac{1}{2}\log(x^2 + s^2) + (-q + (1 \pm 2k\text{II}))\sqrt{-1} \text{ where } x < 0,$$

using the series noted earlier as essential; the convergence is also discussed.

Using $\log(x + s\sqrt{-1}) = \log x + \log(1 + \frac{s\sqrt{-1}}{x})$ and

$$\log(x + s\sqrt{-1}) = \log s\sqrt{-1} + \log(1 + \frac{x}{s\sqrt{-1}}) \text{ he obtains}$$

$$\log(-1) = 2(p + q)\sqrt{-1} \quad \text{where} \quad q = \frac{s}{x} - \frac{1s^3}{3x^3} + \text{ etc}$$

$$\text{and} \quad p = \frac{x}{s} - \frac{1x^3}{3s^3} + \text{ etc}$$

and if $x = s$, $q = p = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ etc

so $\log(-1) = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{ etc })\sqrt{-1}$

If x is taken = 0, $q = \infty - \frac{1}{3}\infty^3 + \frac{1}{5}\infty^5 -$ etc and $p = 0$

from this $\log(-1) = 2(\infty - \frac{1}{3}\infty^3 + \frac{1}{5}\infty^5 - \text{ etc })\sqrt{-1}$ thus giving two different values for $\log(-1)$. The following results are eventually obtained (p.46) :

$$\log(\sqrt{-1}) = \tfrac{1}{2}\text{II}\sqrt{-1} \text{ and } \log(-\sqrt{-1}) = -\tfrac{1}{2}\text{II}\sqrt{-1} = -\log(\sqrt{-1})$$

but $\log(-\sqrt{-1}) = \log(-1) + \frac{1}{2}\log(-1) = \frac{3}{2}\log(-1)$ and $\log(\sqrt{-1}) = \frac{1}{2}\log(-1)$

so $\frac{3}{2}\log(-1) = -\frac{1}{2}\log(-1)$ and this gives $\log(-1) = 0$.

The flaw in this argument, says the author, is that $\log(-\sqrt{-1})$ should be taken as $\log(\sqrt{-1}) - \log(-1)$ and not as $\log(\sqrt{-1}) + \log(-1)$. If this alteration is made, then the result obtained is $-\frac{1}{2}\log(-1) = -\frac{1}{2}\log(-1)$, a correct identity. The formulae for imaginary logarithms are then used to find a value for II or $\frac{\log(-1)}{\sqrt{-1}}$. The next point is that it is possible to deduce from $\log(-1) = \text{II}\sqrt{-1}$ that $\log(-1) = (2k+1)\text{II}\sqrt{-1}$ or $\text{II}\sqrt{-1}/(2k+1)$, but there is also an 'infinity of others'. Likewise it is possible to deduce that $\log(-1) = 2\text{II}\sqrt{-1}/\pm(2k+1)$, where k is an integer (pp.46-49).

The adventurous and ingenious use made of the result $\log(-1) = \text{II}\sqrt{-1}$ by de Missery is just one example of the way in which continental mathematicians had taken the lead over British ones at this time. De Missery was able to make use of it, whereas Playfair referred to it as 'wholly insignificant'. However, in his work on series and manipulation of ∞, de Missery seems to show no knowledge of convergence or concern over meaning. In this he was less advanced than Waring.

De Missery next explores the exponential function $e^{z\sqrt{-1}}$ (p.50), which he says is to be developed as a series. The function he starts with is $e^{z\sqrt{-1}} = fz + \sqrt{-1}f'z$, and he states in a footnote (p.69) that he intends fz and f'z to be cos z and sin z. Here he says that in another memoir he shows that if z is the arc of a circle of raduis one, II is the demicircumference, fz is the cosine, f'z the sine, f''z the cotangent, f'''z the tangent, $f^{\text{IV}}z$ the secant and $f^{\text{V}}z$ the cosecant. This is difficult to follow on the basis that $f''z = (f'z)'$, $f'''z = (f''z)'$ etc, which is evidently not the case. De Missery's own result $(fz)^2 + (f'z)^2 = 1$ is not applicable to f'z and f''z etc, and similarly for his results $f''z = fz/f'z$ and $f'''z = f'z/fz$ etc (pp.50-53). These are obtained by inspection of the results given above. It appears that fz, f'z, f''z etc are unrelated defined functions, and would be better designated fz, gz, hz etc. f'z is not the derivative of fz, if this were the case, f'z, f''z etc would be -sin z, -cos z etc. De Missery's whole thesis is that he is not going to introduce differentiation. However, if f'z is meant to be the derivative of fz, introduced as a defined function, the lack of the '-' sign is an error. If differentiation was intended, the starting point should have been $e^{z\sqrt{-1}} = fz - \sqrt{-1}f'z$, and had this been used no doubt many useful results would have been verified. The results obtained appear to be inconsistent, but to be sure of the line being taken it would be necessary to trace the 'other memoir'. If the functions are unrelated, this is a weaker line of thought than it appears. There is at least a lack of clarity, if not an inconsistency. The use of " ' " at the end of the 18th Century for any function, 'derived' or otherwise, causes much confusion for modern readers.

De Missery takes $e^{z\sqrt{-1}} = fz + \sqrt{-1}f'z$

and $e^{-z\sqrt{-1}} = fz - \sqrt{-1}f'z$, and solving them

together obtains $fz = (e^{z\sqrt{-1}} + e^{-z\sqrt{-1}})/2$

and $f'z = (e^{z\sqrt{-1}} - e^{-z\sqrt{-1}})/2\sqrt{-1}$ (pp.50-52).

He then obtains the relationship $(fz)^2 + (f'z)^2 = 1$, and calculates values for fz, f'z etc when $z = 0, \Pi, 2\Pi, \Pi/2, \pm 2k\Pi, \pm(2k+1)\Pi$ etc, and other properties, such as their signs in various ranges. He next takes $f''z = fz/f'z$ and $f'''z = f'z/fz$ giving no reasons. These assumptions are based on the values of fz, f'z etc which lead inevitably back to the same results previously given for f''z etc.

The most important aspect of this work is not the possible inconsistency of some of the results, but the confidence with which the manipulation of complex numbers was undertaken. It is assumed that complex numbers behave in the same way as reals. This assumption was being stated as a rule by which to explore the properties of complex numbers from this time, particularly by Peacock and de Morgan. De Missery helped to raise their status to that of numbers subject to the usual algebraic manipulations. De Missery's book exemplifies the sophisticated level which had been reached by the beginning of the 19th Century in the assimilation of complex numbers into mathematics. The rules are known and their nature is not questioned or discussed. The interconnections that had now been made with logarithmic, trigonometric and exponential functions and with the calculus were becoming widely known and increasingly used.

Jean-Robert Argand 1768-1822

The single most important and innovative contribution to complex number theory during the whole of the period under consideration was undoubtedly Argand's <u>Essai sur une manière de représenter les quantités imaginaires</u>, published initially as a pamphlet in 1806, and later in Gergonne's <u>Annales de Mathématiques Pures et Appliquées</u>, Paris in 1814. The paper of 1797 (published 1799) by the Norwegian born Caspar Wessel, contained a similar idea for the geometrical representation of complex numbers, but because it remained in obscurity for a century its influence was small. Little information is available about Argand. He lived in Paris, working as a book seller, and the <u>Essai</u> and some related papers were almost his only contribution to mathematics. Neither Wessel nor Argand were professional mathematicians (Wessel was certainly self-taught), and it is salutory that it took the amateur Argand to point out in clear terms the woolly thinking enshrined in the number system nomenclature of his time. Argand's paper was not immediately influential, but his ideas were taken up by Gauss, Cauchy and Hamilton, and are still of great importance today. Gauss and Argand, working at about the same time, appear to have been the first mathematicians to have made serious criticisms of the names in use for certain number categories. Gauss introduced the word 'complex', Argand produced his diagram and a new and logical notation.

The pamphlet of 1806, and its authorship, were almost overlooked by the mathematical establishment. The preface by J. Houël in the English translation of the <u>Essai</u> describes what happened (1) :

> 'Français, an artillery officer at Metz, sent to the Fditor of the Annales [Joseph Diaz Gergonne] the outline of a theory whose germ he had found in a letter written to his brother by Legendre, the latter having obtained it from another author whose name he did not give. This article came to the notice of Argand, who immediately wrote Gergonne a note in which he made himself known as the author of the work cited in Legendre's letter, and in which he gave a complete summary of his pamphlet of 1806. This double publication gave rise to a discussion in the Annales, in which Français, Gergonne and Servois took part, closing with a remarkable article, in which Argand explained more satisfactorily certain points in his theory.'

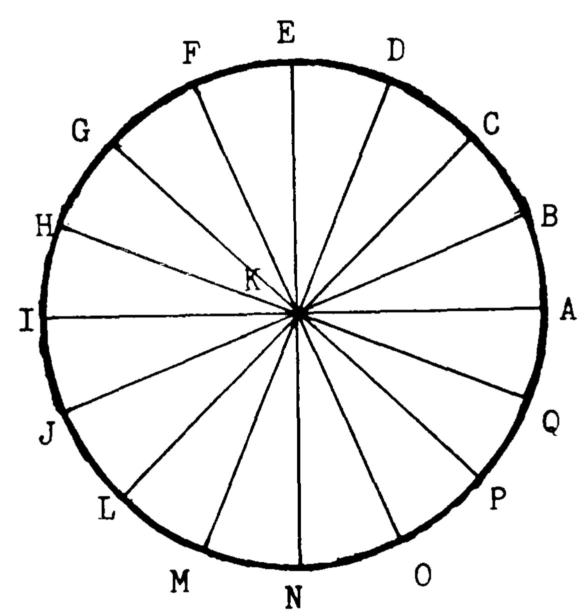
(1) Argand, <u>Imaginary Quantities</u> , p.v (preface)

The <u>Essai</u> begins with a model for negative numbers, which must be extended beyond zero on the number line . This model had not been obvious to all mathematicians (see Wallis and Euler above). Argand gives an example in which objects are weighed using a balance; the principle of moments is used in which distance is separate from direction. He defines $\sqrt{-1}$ in terms of the geometric mean between -1 and +1 in the relationship +1: x :: x: -1. He says (pp.23-24) :

> '. . . as the quantity which was imaginary [negative] when applied to certain magnitudes, became real when to the idea of <u>absolute number</u> we added that of <u>direction</u>, may it not be possible to treat this quantity, which is regarded as imaginary [imaginary], because we cannot assign it a place in the scale of positive and negative quantities, with the same success ? On reflection this has seemed possible, provided we can devise a kind of quantity to which we may apply the idea of direction, so that having chosen two opposite directions, one for positive and one for negative values, there shall exist a third - such that the positive direction shall stand in the same relation to it that the latter does to the negative.'

Argand gives the diagram shown, and his description uses the vector concept although the word 'vector' is not used (pp.24-25) :

> 'For the direction of $\overline{KA}$ to that of $\overline{KE}$, is as the latter to that of $\overline{KI}$. Moreover we see that this same condition is equally met by $\overline{KN}$, as well as $\overline{KE}$, these two last quantities being related to each other as +1 and -1. They are, therefore, what is ordinarily expressed by $+\sqrt{-1}$ and $-\sqrt{-1}$. In an analogous manner we may insert other mean proportionals between the quantities just considered . . . Similarly we might insert a greater number of mean proportionals between two given quantities . . . '



Argand compares $\overline{KA}$ and $\overline{AK}$ with two equal and opposite forces eliminating each other, a concept well-known to mathematicians. He refers to vectors as 'directed lines' and scalars as 'absolute lines'. The diagram emphasises strongly the uniformity in the nature of numbers, whether real or complex, positve or negative. This was the first time such a clear demonstration had been given. Numbers are all represented by lines, and the only difference between those for reals and those for imaginaries lies in their directions. The circular form of the diagram and the method of finding further mean proportionals between two numbers by equal sub-divisions of the angle between the two lines representing them, gives a direct illustration of de Moivre's theorem (with which Argand was familiar). He says he got his initial ideas for representing numbers in a meaningful way from consideration of the inappropriate and illogical names in common use for certain number categories. The numbers themselves are not actually absurd, impossible or imaginary since meaningful results can be obtained from their use. Argand does not use Gauss's word 'complex' which became widely used somewhat later. Argand's point is that mathematicians should take a more mature and realistic view of the number system, both in nature and in nomenclature. He says (pp.31-32) :

> '. . . every line parallel to the primitive direction is expressed by a real number, those perpendicular to it are expressed by imaginaries of the form $\pm a\sqrt{-1}$, and those having other directions are of the form $\pm a \pm b\sqrt{-1}$, and are composed of a real and imaginary part. But these lines are quantities just as real as the positive unit; they are derived from it by the association of the idea of direction with that of magnitude, and are in this respect like the negative line, which has no imaginary signification. The terms <u>real</u> and <u>imaginary</u> do not therefore accord with the above exposition. It is needless to remark that the expressions <u>impossible</u> and <u>absurd</u>, sometimes met with, are still less appropriate. The use of these terms in the exact sciences in any other sense than that of <u>not true</u> is perhaps surprising. An absurd quantity would be one whose existence involved the truth of a false proposition . . .but the results obtained from the use of the so-called imaginaries are in all respects conformable to those derived from reasonings in which only real quantities appear. We might thus foresee the impropriety of a nomenclature which classifies truly absurd quantities and the even roots of negative quantities together, and it was a consciousness of this impropriety which first gave rise to the ideas developed in this essay. It is thus that we are led to a new nomenclature.'

These remarks are very scathing but eminently sensible. Argand does not put forward any new names in place of the unsatisfactory ones but shows much flexibility in his use of, for instance, 'imaginary', which can mean negative, complex or imaginary. However he makes another fundamental advance towards unification of the number system with his suggestions (pp.35-36) for a new notation and an operational approach. He assigns numbers to the four operators as follows : $\sqrt{-1}$ or $\sim = 1$, $- = 2$, $-\sqrt{-1}$ or $\backsim = 3$, $+ = 4$; a straight line counts as two and a curved one as one. To find the symbol for any product, add the numbers corresponding to the signs and subtract fours to obtain a number from 1 to 4. This is the number corresponding to the correct sign. For division the numbers are subtracted. This operational algebraic notation for treating $\sqrt{-1}$ parallels the operational geometric approach of the diagram. Unfortunately this new system was never taken up, it might have helped Hutton to clarify his ideas, if he could have accepted it(1).

Using the circular diagram, Argand gives the construction for multiplying two directed lines and notes that division is the inverse process (p. 36). Other rules and consequences are given, including the product of vectors not measured from the origin, and factorisation of the binomials $x^n + 1$ and $x^n - 1$ in terms of cosines. He uses $\cos na \sim \sin na = (\cos a \sim \sin a)^n$ (de Moivre's theorem), to obtain the series $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ and $x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + \ldots$. The new notation and the vector method are combined to obtain the standard trigonometrical relationships, and the diagram, with suitable arcs, is used (pp.50-52) to obtain series for $\log(1 + x)$ and $\log(1 + z)/(1 - z)$. About polynomials he says (p.79) :

> '. . . every polynomial of the form $x^n + ax^{n-1} + bx^{n-2} + \ldots + fx + g$ is decomposable into factors $x + \alpha$ of the first degree. It is to be noticed that a, b, . . . g are not necessarily reals . . .'

Argand obtains this result from the addition and multiplication of directed lines and is one of only a few writers to consider polynomials with non-integral coefficients.

(1) Hutton's Dictionary (1815) contains no entry under "Argand".

Argand concludes his Essai with these remarks (p.82) :

> '. . the method of directed lines as an instrument of research, whose use is advantageous in certain cases, because geometric constructions offer, as it were, a picture to the eye which facilitates purely intellectual operations. Moreover it is always possible to translate the demonstrations founded on this method into ordinary language.'

Argand is suggesting, quite rightly, that a visual representation of the kind he has given is of help when seeking new developments. He has demonstrated its versatility by using his methods to derive many known series and theorems, although no new results are obtained. In this Essai , Argand is the first to treat complex numbers on a truly equal footing with reals.

Français had written in the Annales describing the new 'geometry of position' and giving a notation in which $1_{\pm\frac{\pi}{2}}$ represents one unit in a direction perpendicular to the real number line(1). Among other ideas, he suggests that this system makes sense of the 'symbolic and mysterious equation' $\frac{\Pi}{2}\sqrt{-1} = \log(\sqrt{-1})$, has applications to circular arcs and the roots of unity, uniting them all in one theory. Argand had responded to this letter, revealing his identity(2) and his work was reprinted in the Annales in 1814. He says that 'direction' is to be preferred to 'position', because $\overline{AB} \neq \overline{BA}$, and uses the word 'module' (modulus), for the first time . Argand also introduces yet another notation, an index notation in which $1^{\frac{1}{4}}$ means $\sqrt{-1}$, that is one unit at $\frac{1}{4}$ of a complete revolution from +1.

Using this notation Argand tries to place numbers such as $(\sqrt{-1})^{\cos\mu + \sqrt{-1}\sin\mu}$ on the diagram, concluding that these vectors must be perpendicular to $\overline{KA}$. He says that these numbers are represented round a circle centre K perpendicular to IA(3), with the modulus determining the distance from K and μ the direction. He admits that there have been demonstrations tending to show that $(a + b\sqrt{-1})^{m + n\sqrt{-1}}$ can be reduced to the form $p + q\sqrt{-1}$, so they should be represented in the original plane. But these demonstrations involve development in series and p and q have not been shown to be finite. Argand says that they are infinite when they represent imaginaries, an idea first suggested by Newton. He says that a number $p + q\sqrt{-1}$ or a_b can become infinite if

(1) Français, Annales , 4(1813-14), 65

(2) Argand, Annales , 4(1813-14), 133-147

(3) Argand, Annales , 4(1813-14), 145 : this new circle takes these numbers into a third dimension

it can be expressed in the form a_{b_c}, and stresses the need to verify the existence of a heirarchy such as a, a_b, a_{b_c} etc. Argand expresses himself uncertain about the nature of the logarithms of imaginaries, he says there is 'a cloud on the spirit'.

Français then wrote to say that as real angles are found in the x,y plane, it is reasonable to expect imaginary angles in the perpendicular plane, but he is not satisfied that two dimensions are insufficient and points out that three dimensions require three coordinates[1]. This point is not the same as that of Argand, who was taking an imaginary modulus and a real angle. Français says that it has been shown that numbers like $(a+b\sqrt{-1})^{m+n\sqrt{-1}}$ reduce to $p+q\sqrt{-1}$, and so must lie in the x,y plane.

Argand questions the rigour of his system[2], moreover it should not only be right, but simple and brief. He also questions the rigour of Euler's proof that $(\sqrt{-1})^{\sqrt{-1}}$ is real, and his work on series for e^z and the formula $e^z = \cos z + \sqrt{-1}\sin z$, where z is complex. Argand is not claiming that his own work is any more correct than that of Euler, only that neither have been proved rigorously, each has only been shown to lead to no inconsistency. He makes the very interesting point that if all the numbers lie in the x,y plane, what can there be that is represented on the perpendicular plane ? He then discusses the relationship $1 : \sqrt{-1} :: \sqrt{-1} : -1$. Servois has expressed scepticism about the mean proportional method, and doubt about the usefulness of directed lines on the grounds that not everyone is able to use them.

For his ideas Argand claims simplicity and ease of application He describes the proofs for the fundamental theorem of algebra as either relying on complex numbers or on development in series which are non-rigorous as they have not been shown to involve only real quantities. The problem is not that the theorem is not true, the problem is the proof. He says that concrete quantities can always represent abstract numbers, but abstract numbers, such as infinitesimals and complex numbers, cannot always represent concrete quantities. This is an argument against Argand's system and he defends infinitesimals by the definition of a limiting value. However Argand returns to the simplicity of his methods and describes in some detail how he obtains sums, products, etc

(1) Français, <u>Annales</u> , 4(1813-14), 222-27

(2) Argand, <u>Annales</u> , 5(1814-15), 197-209

of complex numbers, in terms of their moduli and angles. He claims that his simpler methods must constitute a gain, and compares his directed lines favourably with Lagrangian analysis.

Argand made several other minor contributions to the *Annales*, mostly solutions to geometrical problems. Apart from the brilliant *Essai*, he made no other original contribution to mathematics. Although he was mistaken about the need for a third dimension to his diagram, this work constituted a great reform and simplification of the number system. The work of Gauss and Argand marks the beginning of the clear understanding and proper description of the number system, and the place of complex numbers in it.

Chapter V

The Early 19th Century

The picture of the number system at the beginning of the 19th Century was one in which irrationals were acceptable, negatives were accepted by most mathematicians and complex numbers accepted on the basis of their usefulness and consistency with the reals. The most marked change during the previous two centuries was the advancement of complex numbers from 'useless' to 'useful'. Results, such as the number of roots in a polynomial, had been obtained using them, which could be verified by other means. A powerful visual representation had been given which enabled complex numbers to be constructed geometrically, and which demonstrated graphically their behaviour under algebraic operations. Argand took an operational approach in which '−' is represented by an anticlockwise rotation of 180° and '$\sqrt{-}$' by one of 90°, applied to directed lines. Wessel's representation was similar, but of a more static Euclidean kind. In his paper of 1797 he used vectors and the triangle rule for vector addition, but his ideas were not influential during the 19th Century as his work was overlooked until published in French in 1897. Cotes, de Moivre and Euler were among many who already thought of complex numbers as points in the (Cartesian) plane, but at an intuitive rather than rigorous level.

At this time Gauss was starting to use a number couple notation (a,b), for complex numbers. He used complex numbers in proofs of the fundamental theorem of algebra, seeing them as represented by points in the Cartesian plane. In 1811 he described his idea in a letter to Bessel, in which he says that $a + \sqrt{-1}b$ can be represented by (a,b) (1). The two elements are real numbers taken as an ordered pair, with algebraic rules of combination, from which has been eliminated the symbol $\sqrt{-1}$. By 1831 Gauss had published his description of complex numbers as

(1) Kline, p. 631

number pairs, that is points rather than vectors, with geometric demonstrations for addition and multiplication. Like Argand, he saw the need for new names for number categories, and advocated the words direct, inverse and lateral for positive, negative and imaginary. These excellent suggestions were not taken up, but his less satisfactory 'complex' eventually became universal. In spite of his apparently enlightened attitude, Gauss did not take easily to complex numbers. He regards negatives as validated by the success of results obtained by using them over a long period, and complex numbers as 'still rather tolerated than fully naturalised . . . an empty play upon symbols'(1). Gauss made many remarks showing his lack of confidence in complex numbers, his diffidence contrasts strongly with the certainty of Argand.

The innovations of Argand and Gauss constituted important steps towards clear understanding and definition of the number system later in the 19th Century. The geometrical basis for the revolutionary ideas of Argand may account for the slowness with which they were taken up; although Gauss's contribution with its algebraic emphasis came considerably later than that of Argand, the complex plane has been known as the Gaussian plane. Both interpretations paralleled the Cartesian co-ordinate system, confirming the logicality of extending the axes in the negative directions. The most obvious benefit of the Argand diagram was that it gave a simple visual means of modelling the number system, but, equally important, was Argand's use of it to verify and demonstrate the rules for adding and multiplying complex numbers. This, therefore, was the point when the supposed ambiguities in their behaviour were removed, and the rules of combination seen to be certain and consistent. Gauss's number couple notation can be used in many ways, for instance to eliminate negatives, rationals or irrationals from the number system. Apparently no writers except Hamilton took such a step. This idea might have enabled Frend, de Morgan etc to overcome their scruples about negatives. It was a fortunate coincidence for the status of complex numbers, that the algebraic interpretation became available as confidence in Euclidean rigour declined.

(1) Tahta, Complex Numbers, p. [8]

The step from the one-dimensionality of the reals to the two dimensions of complex numbers was, in each of the new definitions, the essential innovation, and gave rise inevitably to attempts to generalise to three and more dimensions. The success of new theory and notation is measured not only by whether it is easy to understand and manipulate, but also on whether it facilitates new ways of thinking. In this sense the two dimensional approach to complex numbers was highly successful, as it led Hamilton to generalise to quadruples, and Grassman to n-tuples. These discoveries played an important part in the 19th Century reorientation of ideas about mathematics.

As the ideas of Argand, Gauss and others became better known, the potential acceptability of complex numbers increased. To investigate this it was decided to examine briefly work of three early 19th Century mathematicians, Cauchy, Hamilton and de Morgan.

Cauchy laid the foundations of complex function theory. Although in 1821 he described results obtained using $\sqrt{-1}$ as not making sense unless real and imaginary parts are separately equated, in the same work he was using $\sqrt{-1}$ to get results without employing this technique[(1)]. An example from number theory (not original with Cauchy) is given below. In 1822 he gave a method for integration round a rectangle showing that the integral is independent of the path, in 1825 he considered integration of real functions using complex limits[(2)]. He used complex numbers in many novel ways but his treatment was algebraic. Not only did he not make use of the Argand diagram or complex plane, he does not appear to have used Gauss's number pair method either.

(1) Cauchy, Oeuvres, (2),3,154; (Cours d'Analyse)

(2) Kline, p. 635-36

Cauchy did not see complex numbers as having any other than an abstract meaning. In the Cours d'Analyse of 1821 he considers the expressions $\cos a + \sqrt{-1}\sin a$ and $\cos b + \sqrt{-1}\sin b$, and their product $\cos(a+b) + \sqrt{-1}\sin(a+b)$ (1). He describes them as symbolic expressions which do not represent anything real, but does not suggest anything that they might represent. He finds it strange that the first two can be multiplied to obtain the third. He sees complex expressions as having an important role in containing two pieces of information simultaneously, one in the real part and one in the imaginary. He refers to $\sqrt{-1}$ as a coefficient and to the equating of real and imaginary parts. What is being equated in these methods is actually two pairs of real quantities, one pair having each the coefficient $\sqrt{-1}$. A few pages further on he demonstrates the power of complex numbers to produce results in number theory, when he uses them to prove that the product of two numbers, each of which is the sum of two squares, is itself the sum of two squares in two different ways. For instance $(2^2+1^2)(3^2+2^2) = 4^2+7^2 = 1^2+8^2$, $(5 \times 13 = 65)$. The proof is as follows (2) :

$$(a + ib)(p + iq) = ap - bq + i(aq + pb) \quad \text{and}$$

$$(a - ib)(p - iq) = ap - bq - i(aq + pb)$$

Multiplying gives :

$$(a^2+b^2)(p^2+q^2) = (ap - bq)^2 + (aq + bp)^2 \qquad \text{(i)}$$

in which p and q are interchangeable on the left. Interchanging them on the right gives :

$$(a^2+b^2)(p^2+q^2) = (aq - bp)^2 + (ap + bq)^2 \qquad \text{(ii)}$$

Equations (i) and (ii) are two different ways of decomposing $(a^2+b^2)(p^2+q^2)$ into the sum of two squares.

Cauchy expresses the view that complex numbers are extremely useful in algebra and analysis as well as in number theory. He covers de Moivre's theorem, series for trigonometric functions of complex numbers, roots of complex numbers and other similar topics. He describes symbolic algebra, which may contain imaginaries, as one in which a fixed set of rules are obeyed, but says that that the expressions obtained may be entirely abstract, that is, devoid of meaning.

(1) Cauchy, Oeuvres, (2),3,154

(2) Cauchy, p.159

Although Cauchy had little confidence that complex numbers represented anything meaningful, he made good use of them to obtain important new results. He continued the fruitful work of continental mathematicians on the involvements between complex numbers, the calculus and mathematical functions. British mathematicians who, at this time, were not in the forefront of developments in the calculus, were turning to algebra; in this area is found their main contribution to complex number theory.

The English mathematician George Peacock published the 'principle of the permanence of equivalent forms' in his Treatise on Algebra of 1830. He wrote(1) :

> 'Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true, whatever these symbols denote.'

Peacock had formulated the commutative, associative and distributive laws, as they applied to numbers and to polynomials. His Algebra contained an attempt to describe a formal algebra in abstract terms, tied implicitly to number as it conformed to the rules of number. Peacock's Principle was shattered by Hamilton's non-commutative algebra of quaternions published in 1843, and by the doubly distributive algebraic system of George Boole of 1854, and by other work. The Principle was discredited so soon after its formulation that one must speculate whether the commutative law, for instance, could have been discarded by Hamilton had it not first been pointed out by Peacock.

Alexander Macfarlane wrote in 1916(2) :

> 'When algebra is based on any unidimensional subject, such as time, or a straight line, a difficulty arises in explaining the roots of a quadratic equation when they are imaginary. To get over this difficulty Hamilton invented a theory of algebraic couplets . . . '

Hamilton extended Gauss's work on number pairs. He thought that space and time were indissolubly connected, with geometry being the science of space and algebra that of time. The algebra of quaternions

(1) Peacock, Algebra , p.104

(2) Macfarlane , Lectures , p.42

can be used to transform a three-dimensional vector, but is free of geometrical ideas, depending for its validity on the consistency of the number system.

Hamilton's paper Theory of conjugate functions . . Algebra as the Science of Pure Time was given in 1833 and 1835. Even he could entertain doubts about negatives, he says[1] :

> 'But it requires no peculiar scepticism to doubt, or even to disbelieve , the doctrine of Negatives and Imaginaries . . . '

He debates whether algebra is a science like geometry, with a system of rules, or an art like a language, a system of expression. It seems to be useful only so far as it is applicable. In the discussion he uses the word 'magnitude' as well as 'number', and later refers to 'step-couples' and 'moment-couples', that is locations and vectors. Most of what follows refers to numbers, but evidently Hamilton is bearing in mind both geometric and algebraic approaches.

Hamilton justfies abandonment of $\sqrt{-1}$ in favour of number couples as follows[2] :

> 'In the THEORY OF SINGLE NUMBERS, the symbol $\sqrt{-1}$ is absurd, and denotes an IMPOSSIBLE EXTRACTION, or a merely IMAGINARY NUMBER; but in the THEORY OF COUPLES, the same symbol $\sqrt{-1}$ is significant, and denotes a POSSIBLE EXTRACTION, or a REAL COUPLE, namely . . . the principal square root of the couple (-1,0). In the latter theory, therefore, though not in the former, this sign $\sqrt{-1}$ may properly be employed; and we may write, if we choose, for any couple (a_1,a_2) whatever, $(a_1,a_2) = a_1 + a_2\sqrt{-1}$.'

This seems to bring Hamilton back to the 'absurd' symbol $\sqrt{-1}$, the difference is that the symbol is not to be used in number manipulations, but only as an alternative way of expressing a number pair. He does not say that imaginary numbers are absurd, only the symbol $\sqrt{-1}$, but an imaginary number can be represented by any symbolism one chooses, and the symbol cannot be any more or any less absurd than the concept it represents. Hamilton is not as whole-hearted as Argand in rejecting such words as 'absurd' when dealing with the number system.

(1) Hamilton, Mathematical Papers , III, p.4

(2) Hamilton, p.93

Hamilton tried to make the case that his algebra related to time. The view that there can be many algebras, that they are abstract and can relate to many systems but need not be tied to any one, was rapidly gaining ground. In this sense it was not significant that Hamilton related his algebra specifically to time, this would soon be disregarded. A mathematician who was prominent in the move towards the formalisation of algebra as sets of rules was de Morgan, whose views on negative and complex numbers were, in many ways, similar to those of Hamilton.

Augustus de Morgan, Professor of Mathematics at University College London, was an algebraist in the forefront of his field in the 1830's and 1840's. He was son-in-law to William Frend and shared some of Frend's views on negative and complex numbers, although he was not as extreme as Frend. De Morgan was a prolific writer of articles on many subjects and contributed to the Penny Cyclopaedia, which was published in weekly parts by the Society for the Diffusion of Useful Knowledge. It appeared from 1833 to 1837, amounting eventually to twenty-seven volumes. His articles on mathematical topics in this publication refer to recent papers and provide an up-to-date description of the state of algebra in the late 1830's. It can be seen that de Morgan did not have great confidence in his subject, although he went on to develop many new ideas in symbolic algebra. It is appropriate to conclude this study with some extracts from these articles.

There is no entry under 'Complex' but entries under 'Negative and impossible quantities' and 'Operation' consist of the two parts of a single long article and give the writer's views on negative and complex numbers. Like Frend, he calls the arithmetic of negatives an art, but never took the step of eliminating them from algebra. He distinguishes between their use and meaning as follows(1) :

> '. . . a modification of quantity unknown in arithmetic called negative quantity, as distinguished from positive . . . a generalisation of which the use was obvious, but not the meaning . . . [there being an] obvious deficiency of rational explanation which characterised every attempt at their theory.'

He says that algebra is learnt by rules rather than understanding, and verified by the correctness of results, a view that would be unpopular today. He regards positives and negatives as inhabiting two separate worlds, he makes no reference to their continuity or to a number line. To de Morgan, the inclusion of negatives represents the step from arithmetic (a science) to algebra (an art). This has to be done using a set of rules to keep the results obtained for negatives consistent with those for positives. He says (p. 132) :

(1) De Morgan, "Negative and impossible quantities", Penny Cyclopaedia , Vol.16, pp.130-37, (p.130)

'The first step from arithmetic to algebra is made by the following definitions : -
1. Quantities are distinguished into positive and negative which are to be considered of diametrically opposite kinds; and common arithmetical quantities (abstract numbers without signs) are to be considered as positive.
2. The rules of arithmetical algebra are to be applied to the extended algebra, and in all cases in which the latter presents a case unknown in the former, the rule of signs already known in the former must be applied.'

Addition and subtraction are described as 'operations', signed number notation is not used. This limited view taken of negatives contrasts with the sophisticated idea that algebra may be abstract in the sense of being independent of the meanings of quantities involved. But de Morgan seems to have been influenced towards the idea of a symbolic algebra by the symbol $\sqrt{-1}$, a symbol that he regards as virtually meaningless. He says (p. 134) :

> 'In such a case where the meaning of a symbol [$\sqrt{-1}$] is left undetermined. . . if such meaning cannot be given, then the symbol is properly called impossible; if it can be given in more ways than one, it is usually called ambiguous.'

He does not give an example of an ambiguous symbol, he may have been thinking of a square root.

De Morgan's view of negative numbers is not a good foundation for a clear understanding of complex numbers. He says (p. 136) :

> '. . . no result was fit for actual application until the impossible quantities had disappeared.'

In spite of this remark de Morgan goes on to obtain de Moivre's theorem, he deduces expressions for sine and cosine using series, and for the binomial theorem, and demonstrates that any algebraic function of $\sqrt{-1}$ can be reduced to the form $A + B\sqrt{-1}$, so that algebra leads to no more impossibles. He shows that reals have an infinite number of logarithms, covers the roots of unity and uses trigonometric functions to deal with the irreducible case of the cubic. He gives Wallis's mean proportional definition for $\sqrt{-1}$ and describes the Argand diagram. He gives the rotational justification and uses $e^{i\theta} = \cos\theta + i\sin\theta$ to define a unit line at angle θ. He emphasises that numbers on the real axis are only a special case of numbers on the Argand diagram (p. 136) :

> 'For lines measured in that unit line, the extended definitions coincide with the ordinary ones.'

Although de Morgan does not regard $\sqrt{-1}$ as representing anything that exists, and in spite of his serious reservations about negatives, he uses them in deriving many important results. In the same article he makes it clear that it is their usefulness that gives complex numbers their place in mathematics. About Cauchy and Hamilton he says (p. 137) :

> 'Mr. Cauchy and others had previously considered it as merely a symbolical contrivance to express the coexistence of two equations thus $a + b\sqrt{(-1)} = c + d\sqrt{(-1)}$ is a well-known method of implying $a = c$ and $b = d$, both in one equation. The manner in which Sir William Rowan Hamilton has connected this symbol with his system would justify us in saying that, if his science of time were retranslated into a science of magnitude, his explanation of impossible quantities would fall back into the one I have just alluded to.
>
> We are inclined to think that this explanation of algebra with reference to time may finally be admitted as one method of supplying the foundations of the purely symbolical science : but we must confess ourselves not yet sufficiently clear upon the matter in which the symbol $\sqrt{(-1)}$ is connected with its definition, to hazard a positive opinion.'

Although the last remark is somewhat ambiguous, de Morgan seems to be saying that he does not fully understand Hamilton's system, but suggests that his algebra does not only apply to time, but also to magnitude and number.

Similar views are expressed in the second part of this article, under 'Operation' (1). He continues with his ideas for a symbolic system, attributing the first use of symbolism to represent 'directions how to proceed with magnitudes' to Newton and Leibniz in the calculus. Negatives and their square roots are both used as possible elements of a symbolic algebra in which specific meanings need not be attached to all symbols. This is an early algebra that is truly symbolic.

(1) De Morgan, "Operation", Penny Cyclopaedia , Vol.16, pp.442-46

De Morgan still held the same views about negatives and imaginaries, and about algebra as a 'useless' art in 1849, when he published Trigonometry and Double Algebra[1]. 'Double' algebra was that of complex numbers, 'single' algebra involved negatives, and the algebra of positive reals was 'universal arithmetic'. In this book he speaks of the experimental use of the unexplained symbol $\sqrt{-1}$, and 'intelligible results when such things occur', showing that his doubts about complex numbers were, if anything, becoming more serious.

During the period under consideration some mathematicians expressed the view that complex numbers did not represent anything real in the sense that they did not represent anything at all. By the early 19th Century it was generally thought that complex numbers were useful as there was no doubt that they gave valid results. It is surprising that there was still so much uneasiness about negatives. This uneasiness seems particularly to have afflicted English mathematicians, though even Hamilton was sceptical. It was also English mathematicians who, following Frend, were concerned over the status of algebra, wishing to categorise it as either a science or an art. It was more desirable that it should be counted a science, but to some it was debarred by containing complex numbers and to others by containing negatives.

(1) Smith, "De Morgan and the foundations of algebra", pp.9-13

Chapter VI

Summary and conclusion

Absurd	Irrational	Positive
Chimaera	Irreducible	Rational
Complex	Monster	Real
Desperate	Negative	True
False	Ridiculous	Useful
Ficticious	Sophistic	
Figment	Tortures	
Imaginary	Useless	
Impossible		

Collected above are some of the extraordinary terms encountered while researching this study, as having been applied to number subsets or to polynomials or equations, during this period. It can be seen that adverse names considerably outnumber favourable ones. This was the position reached by neglecting to allow nomenclature arising as reluctant steps were taken into new number subsets, to be superceded. The obscure mathematician Argand, with commendable commonsense pointed out the illogicality of some of these adverse terms and proposed a useful new symbolism for $\sqrt{-1}$ etc, which unfortunately was not taken up. It is inevitable that words change their meanings and associations and, to begin with, many of these words would have been merely descriptive without a well-defined technical meaning. Where they were used semi-technically (for instance 'impossible'), the meanings were not particularly precise, and some cases have been mentioned of words having been used with different meanings on different occasions. But the number system is an important and sophisticated structure, and it is very desirable that suitable names should be devised for its subsets.

An attempt has been made to identify a point at which mathematicians ceased to refer to roots as 'impossible' and started to use this word for the problems whose solutions the roots represented. In other words to separate the problem and its properties from the number system and its properties. This attempt has not been notably successful. Saunderson thought of the numbers as impossible, Wallis refers to 'the imaginary roots of impossible equations'(1) and, when giving examples leading to complex answers, refers to the problems rather than the solutions as impossible. Wallis seems to have been ahead of his time in this, Euler reverts to using both 'impossible' and 'imaginary' for numbers, stating in somewhat reluctant terms that complex numbers must be impossible but elsewhere referring to the problems as impossible(2). One of the confusing things about Waring's Meditationes Analyticae is the language used, but he does not use words meaning 'imaginary'. In Hutton's Dictionary 'Impossible' is entered, but the reader is referred to 'Imaginary'. Under 'Root' he says 'impossible or imaginary' indicating that 'imaginary' is preferable, 'impossible' having been added merely for clarity. Under 'Imaginary' Hutton sometimes uses this word and sometimes 'impossible', but he is evidently moving away from the use of 'impossible' for complex numbers. Gauss, who introduced the term complex, did not use the word 'impossible', but de Morgan reverts to it(3) in the Penny Cyclopaedia. Argand points out the general unsatisfactoriness of current nomenclature and uses the word 'imaginary' with the modern meaning; Maclaurin had also usually used this word in the same way. Some caution must be exercised here as neither Waring, Euler, Gauss nor Argand was writing in English, and much of Maclaurin's Algebra was compiled posthumously. There is no clear-cut point after which the word 'impossible' was dropped for a complex or imaginary quantity, but it may be said that its use declined during the early 19th Century.

(1) Wallis, Algebra , p.[v], (preface)

(2) Euler, Algebra (1797), p.64

(3) De Morgan, "Negative and impossible quantities", Penny Cyclopaedia , Vol.16, pp.130-37

This study is in no sense comprehensive, a selection has been made among available sources. Most use has been made of works on algebra as these have been most helpful in providing information on the points being considered (listed in the Introduction). Complex numbers appeared first in algebraic works, it was the algebraists who discovered and described them, and had most to say about their nature. Trigonometry, calculus and analysis were first developed in the real number field, their extension into the complex number field was a subsequent step. Algebraists may be thought of as originators or constructors while writers in trigonometry, calculus and analysis were users and applyers, who accepted and used complex numbers and their rules as described by the algebraists. This division is not clear cut as Euler, for instance, can be placed in both categories.

The fundamental theorem of algebra was widely accepted, though not rigorously proved until after the end of the 18th Century. If the Argand diagram represents geometrical clarification of complex numbers, then Gauss's proofs of the fundamental theorem represent their vindication from an algebraic stand-point. The consequence of this theorem is that roots of all kinds, negative, irrational and complex included, must be summed together. This must imply that these are all entities of the same kind, and that complex numbers are, in fact, numbers. Any mathematician with a sense of pattern in mathematics must have recognised the desirability of this simplification.

Few mathematicians considered polynomials having coefficients that were other than natural numbers. The nature of roots was studied, but the normal assumption is that coefficients (and powers) are not irrational or complex, and in some cases, not even negative. Descartes briefly considered irrational coefficients, mainly as entities to be eliminated, Frend does not even entertain negative ones. Other mathematicians, Euler and Newton for instance, were more interested in non-integral powers than in non-integral multipliers in polynomials. Before their integration into trigonometry etc, complex numbers passed through a phase of being acceptable as roots of equations but not elsewhere.

A recurrent theme has been that number categories have been accepted because they were useful and produced desired results, and not because their introduction was based on any sound theoretical foundation. Girard and Gauss expressed this view about negatives, Vieta about irrationals and Newton, Hutton, Laplace, Lagrange and de Morgan about complex numbers. Pell, Collins and Wallis all expressed the view that complex solutions could be used as an indication or measure of the impossibility of a problem, although there seems to have been no useful attempt to quantify this. Euler and Newton both said that complex answers are needed to cover cases where a problem has no answer, that is, no real answer. It has been mentioned that it was during the period being studied that complex numbers advanced in status from 'useless' to 'useful'. I suggest that this is the single most important factor that has been identified. The new discoveries made about them enhanced and emphasised their usefulness, and the increasingly favourable view of their usefulness gave point and purpose to further investigation of their properties.

It is not uncommon for mathematicians to evade aspects of their subject that they have not understood, Euler being a notable exception. Many mathematicians ignored complex numbers, some avoided them where possible or recommended their avoidance. One reason for avoidance of complex numbers was the occasional published error. Mistakes published by Bombelli and Euler have been mentioned as having had important repercussions. Among writers who have consistently or occasionally avoided negative and complex numbers are Vieta (avoids negative and complex numbers in <u>Arithmetica Speciosa</u>), Oughtred (avoids both in the <u>Clavis</u>), d'Alembert (avoids complex numbers in the <u>Encyclopédie</u>), Hutton (avoids complex numbers in the <u>Course of Mathematics</u> and says in the <u>Dictionary</u> that they should be avoided), Frend (eliminates both from algebra), Gauss (avoids complex numbers as long as possible in his proofs of the fundamental theorem of algebra). Lagrange hardly mentions complex numbers in his <u>Additions</u> to Euler's <u>Algebra</u> in spite of the prominence given them by Euler; and in one of his lectures, when he says that certain roots cannot be rendered independent of imaginary quantities, implies that this would have been desirable(1). It is not unknown for mathematical discoveries that have been made in one way to have a totally different proof devised for them, the usual reason being that the original proof or discovery method was thought to lack credibility. Hutton wrote(2) :

> '. . . the theorems that are sometimes discovered by the use of this symbol [$\sqrt{-1}$] may be demonstrated without it by the inverse operation, or some other way'

Laplace(3) is among other mathematicians who make similar observations.

With hindsight much of the work of 18th and early 19th Century mathematicians on matters related to the number system can be seen as filling in details in a structure that was broadly known. Although it was not realised at the time, all the subsets of the complex number field and their behaviour, were essentially known by the mid-18th Century. When d'Alembert and Euler showed that a complex number raised to a complex power gives another complex number, this meant that the complex number field was known to be closed under the five algebraic operations. There was therefore no need to seek a larger number set, there were no unresolved gaps to fill.

In conclusion, it is necessary to summarise the extent to which the points listed in the Introduction have been resolved.

(1) Lagrange, Lectures , p.87
(2) Hutton, Dictionary (1796), p.147
(3) Kline, p.628

(i) The establishment of the rules of behaviour of complex numbers

Bombelli gave the four rules for negative and complex numbers and, although the first printed version of his Algebra contained errors, these were corrected and it is true to say that the period began with a sound arithmetical foundation. Unfortunately these rules did not spread and gain acceptance as they deserved. I have mentioned Harriot's trial with an incorrect rule for (-) x (-) and Euler's errors in the multiplication of imaginaries. In his Dictionary, Hutton summarises contradictory information in circulation near the end of the period. Although not every mathematician was in such a state of confusion, Gauss for instance, there are grounds for saying that uncertainty about the four rules for complex numbers was greater at the end of the period than at the beginning. However, as more mathematicians were using complex numbers at the end than at the beginning, the number using them effectively would also have been greater. The properties of conjugates were given correctly by Bombelli, and first came into prominence in the solution of the cubic. These properties seem to have been well understood in spite of their unexpectedness, and did not become the subject of controversy. Newton first gave a rule for the number of complex roots in a polynomial, but it was not justified until the mid-19th Century. Towards the end of the period the rules for powers and roots were given by d'Alembert and Euler, Euler being the first to give a value for $\sqrt{-1}^{\sqrt{-1}}$. All these algebraic rules were used when complex numbers were incorporated into trigonometry, calculus and analysis. The Argand diagram gave a geometrical demonstration of the rules and it could be used to show that they were correct. In extending the real number field to include complex numbers, the criterion was that the rules should be such as to give the accepted real number results when restricted to reals. This view was stated explicitly by Euler, Peacock, de Morgan etc. The process was still that of extension and synthesis in the 18th Century; analysis of the complex number field into its component subsets did not start until well on in the 19th Century.

(ii) The usage of complex numbers

Complex numbers, which came forcibly to the attention of mathematicians through the solution to the cubic, soon had to be accepted as roots of quadratics. Many mathematicians (Euler, Descartes), accepted the fundamental theorem of algebra. Proofs were put forward by Gauss, the later, most satisfactory ones relying on complex numbers in the proof. One of the benefits of this theorem was the great simplification brought to the algebra of polynomials, and it demonstrated that complex roots had an essential part to play. During the 18th Century it had also become clear that complex numbers were essential to complete the algebra of logarithmic, exponential and trigonometric functions. So complex numbers were known to be vital to complete understanding of several different branches of mathematics at the beginning of the 19th Century. Also, by this time, mathematicians had started to extend calculus and analysis techniques to complex numbers as a wider branch of mathematics in its own right, of which real calculus and analysis formed a part. The powerful technique of separating a complex function into its real and imaginary parts was used to solve various problems, and was especially useful in real integration. In the solution of problems, complex answers were seen by some mathematicians as a mathematical means of recognising that a problem was impossible and even of assessing the degree of impossibility (Collins, Newton, Euler etc). I have suggested that the view of complex numbers as useful rather than useless was the single most important advance during this period.

(iii) Manipulation of symbols

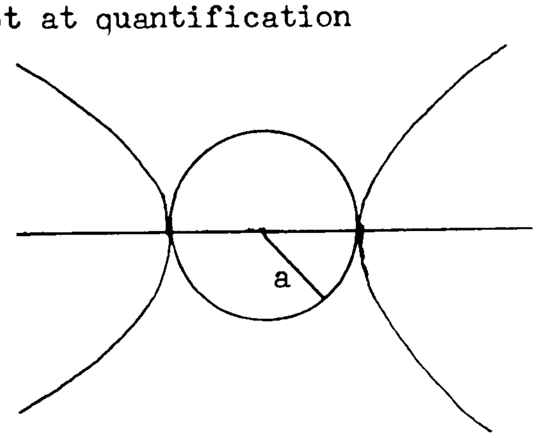
Many mathematicians were able to use the symbol $\sqrt{-1}$ together with its rules of behaviour to great effect; this spite of the fact that some of them were expressing doubts about its nature. Most prolific was Euler but Cotes, d'Alembert, de Moivre, Laplace and Leibniz were also important. The lack of definitions and visual representations for complex numbers makes the successes achieved the more remarkable. The emphasis on verification of such results by other means (Lagrange, Maclaurin), and the evasion of complex numbers (Gauss, Frend, Hutton etc), are hardly surprising. Such verification might have prevented some of the slips mentioned (Euler, Playfair) and rendered results compatible with those already known. Euler's symbol 'i', Gauss's number couples and Argand's $\sim$ and $\sim\!\!\!|$ were all potentially useful for clarification of ideas. Symbol manipulation enabled the relationships between logarithmic, exponential, trigonometric and complex functions to be discovered, even if in somewhat mechanical ways, and calculus methods to be applied to them. Advances made in this superficial way could not be made more insightful until the number system was put on a sounder theoretical basis. Symbol manipulation is an important process for mathematical advancement under these circumstances. The question of detailed interaction between $\sqrt{-1}$ and other totally different symbols, such as D, is the one on which least progress has been made in this study. Rather few examples have been encountered, partly because the use of such symbolism for advanced concepts had not come into general use by the early 19th Century. This point is the one which might most repay further research, particularly towards the end of the period.

(iv) Views of mathematicians on the nature of complex numbers

The question of the metaphysical nature of complex numbers has been the most interesting one. This constituted a great difficulty and was undoubtedly a reason why some mathematicians avoided them, some cases of this have been described. Words attached to the subsets of the complex number field, and the adverse nature of most of them have been discussed. It is easy to understand the inevitable effect of the implied attitudes contained in them. Even the favourable terms such as real or true carry the implication that somewhere there is some non-real, untrue aspect or entity from which they must be distinguished, and it is difficult to see how these problems can ever be resolved. In the metaphysical sense the answer is that complex numbers are by nature two-dimensional or two-element numbers, this answer was not given until the early 19th Century. They first arose in the solution of equations which were expected to give numerical roots, but did not always appear to do so. The properties of conjugates and their strange ability to extinguish imaginary parts when added or multiplied, were known during the whole of the period. In the mathematical sense, Wallis gave geometrical and arithmetical interpretations, and Argand gave an improved geometrical interpretation. The algebraic number pair interpretation given by Gauss gave further insight, and this idea proved fruitful to later algebraists. By the early 19th Century, complex numbers had been interpreted algebraically, arithmetically and geometrically, and many words had been used to describe them. Some of these terms, such as 'impossible', tended to place these quantities not only outside mathematics but outside reality itself. Algebraists have been most inclined to discuss in print the nature of complex numbers, and Euler expressed most openly the doubts shared by many writers of his time. The eccentric English mathematician Frend took scepticism to the greatest lengths when eliminating even negatives from his Algebra, a book whose contents would not have seemed strange to mathematicians of a millennium earlier.

(v) Representations and models for complex numbers

Mathematicians' earliest encounter with complex numbers involved the roots of equations, an equation being thought of as representing an actual concrete problem. Complex roots arising in a quadratic could not be represented in the diagrams of Al-Khowarismi because a negative area was involved, or in the Cartesian plane because no real intersections could be found. Neither could the roots be interpreted in a concrete way in terms of the solution to the problem, because complex roots to the equation meant that there was no real solution to the problem. This meant that mathematicians had to look elsewhere for a representation or model, and they found it very difficult to know where to look. Wallis's diagram has been mentioned, also his attempt to gain more concrete insight by the study of problems giving complex answers. Wallis must be acknowledged as the writer making most contribution on this question up to the end of the 18th Century. The notion that the imaginariness of the solution measured the degree of impossibility of the problem was fairly widespread, but no successful attempt at quantification has been found. In 1768, W.J.G. Karsten produced a diagram(1) which showed the many logarithms of a real or complex quantity represented by the circle whose ordinates are the imaginary ordinates of a hyperbola. As shown in the diagram, this circle is the one whose centre lies on the axes of symmetry and which touches the two branches. For the hyperbola $y^2 = x^2 - a^2$, and for the circle $y^2 = a^2 - x^2$. This diagram was not capable of much generalisation. The first break-through came with the Wessel/Argand diagram and the complex plane of Gauss. These must constitute by far the most important step in providing a representation for complex numbers.



(1) Cajori, "Historical note on the Graphic Representation of Imaginaries before the time of Wessel", Amer.Math.Monthly, 19(1912), 167-71, (p.170)

(vi) The status of complex numbers and attitudes to them

The attitudes of mathematicians to the number system can be inferred from the list of terms given above. The unfavourable terms not only express lack of confidence in the number system by those who devised them, but perpetuation of this attitude in those who continued to use them. Succeeding mathematicians, and others, must have absorbed the impression that the number system contains some very obscure and difficult elements, and this is especially so for the complex numbers. As complex numbers were seen over the period as increasingly useful, attitudes became more favourable, and eventually Argand and Gauss were able to make some valuable suggestions for reform. By the early 19th Century complex numbers were known to be vital to several branches of mathematics, and the fundamental theorem of algebra made it essential to regard complex roots as numbers with status similar to that of other roots. The Argand diagram gave a geometrical interpretation from which '$\sqrt{-1}$' could be eliminated, and number couples did the same for an algebraic approach. However, the impression remains that many mathematicians were still very unsure about complex numbers, and some examples have been given. They had been forced into the formalist position of having to accept an undefined set of numbers with known rules of behaviour, but whose nature was not thought to be well understood. The formalist stance was not to be described until well into the 19th Century, and the dissatisfaction with this state of affairs is clear. The fact that no other number categories had been defined was not noticed because it was possible to feel an intuitive comprehension of these; this may be attributed to the fact that there were plenty of simple concrete models and diagrammatic representations for them. Mathematicians have been mentioned who have avoided complex numbers where possible, others who totally ignored them and a few who have made ambiguous statements about them. Some, like Wallis, were able to deal with them in a seemingly cool and confident way, whereas others, such as Euler, expressed doubts but nevertheless manipulated them effectively to obtain new and useful results. Hutton summarised what he saw as an area of confusion in the rules of operation. The many discoveries made, and their recognised usefulness, were advances which should have consolidated confidence in complex numbers by the early 19th Century. It is remarkable that they do not seem to have done so. Even de Morgan, a pioneer of symbolic algebra, had little confidence in either complex or negative numbers. It was this very lack of confidence which led him to devise a symbolic system, in which not all terms were necessarily defined. Attitudes are often difficult to assess and much has to be inferred, few writers have been willing to express their opinions openly. Those who have done so have provided some fascinating insights.

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Appendix I

J J Sylvester 1814-1897

Sylvester held several University posts on both sides of the Atlantic, but also spent sixteen years as Professor of Mathematics at the Royal Military Academy Woolwich (see Hutton). It was during this period, in 1864 and 1865, that he produced three papers[1] on the proof of Newton's rule. In the first he expresses admiration for Newton's discovery and points out that many other mathematicians (Maclaurin, Waring, Euler) had tried unsuccessfully to find a proof. He dismisses the quadratic and cubic as trivial and gives proofs for the quartic and quintic. The quartic is expressed homogeneously in x and y and these are given infinitesimal increments. These are used to prove the rule. A different, graphical method is used for the quintic. This paper was long, over 100 pages, it did not provide a general method. The second paper of 1865 dealt with the sextic and above, it used the sign of the second differential in the neighbourhood of the roots. This paper did not provide a proof, and consisted of only two pages.

It was in the third paper of 1865 that the first satisfactory general proof was given. This paper was the syllabus of a lecture given at King's College London and to the Mathematical Society of London when Augustus de Morgan was in the chair[2]. In this method, which is algebraic, the polynomial is written in the form $fx = a_0x^n + na_1x^{n-1} + \frac{1}{2}n(n-1)a_2x^{n-2} + \cdot \cdot \cdot \cdot + na_{n-1}x + a_n$ with $A_{n-1} = a_{n-1}^2 - a_{n-2}a_n$ (the differences used by Newton). He then considers $\left.\begin{matrix} a_r \\ A_r \end{matrix}\right\}$ an associated couple of elements, and $\left.\begin{matrix} a_r, & a_{r+1} \\ A_r, & A_{r+1} \end{matrix}\right\}$ an

(1) Sylvester, <u>Mathematical Papers</u>, II, pp.376-479;493-94;498-513

(2) Sylvester, II, 498-513

associated couple of successions. He then considers the permanence or variation of signs in successive associated succession couples, taking all cases, noting the effects of these on the roots, also the effects of the omission of terms, in a close and detailed analysis. He traces

> '. . . the law of change in the number of double permanences . . . as x increases continuously. No change can take place except at the instant when one or more of terms in the inferior or superior series, or in both simultaneously become zero . . .
>
> 'Thus for a single vanishing of an intermediate term in the upper or lower series double permanences may be gained as x continually increases but never lost.'

The same is true for the lower series. The series referred to are the terms of the polynomial and the Newtonian fractions added above. The law of change in double permanences has to be laboriously verified as x changes continuously, to check that the number can only change when a term in one of the series is zero.

Unfortunately we do not know how Newton discovered his rule, or how he justified it. The successful proof of Sylvester is complicated and difficult to describe. It does not produce the rule and if Newton had a proof of this kind, he must have discovered the rule in some other way. Newton may have just guessed the rule from simple cases or may not have thought it worthwhile to write out the details of such a proof.

Appendix II

Some suggestions for further research

While working on this study it has been impossible to overlook the fact that there is a great deal of material still to be investigated. It is suggested that the following may be of most importance.

Work in algebra of :

John Napier (1550-1617)
Thomas Harriot (1560-1621)
Alexis-Claude Clairaut (1713-1765)
Carl Friedrich Gauss (1777-1855)

Work in trigonometry, calculus and analysis of :

Roger Cotes (1682-1716)
Abraham de Moivre (1667-1754)
Louis Arbogast (1759-1803)

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January 1985