A Logic of Isolation

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Abstract. In the vein of recent work that provides non-normal modal interpretations of various topological operators, this paper proposes a modal logic for a spatial isolation operator. Focussing initially on neighborhood systems, we prove several characterization results, demonstrating the adequacy of the interpretation and highlighting certain semantic insensitivities that result from the relative expressive weakness of the isolation operator. We then transition to the topological setting, proving a result for discrete spaces.

Keywords: Topological Semantics · Neighborhood Systems · Isolation

1 Introduction

Topological interpretations provided some of the earliest semantics for modal logics (e.g., [12], [7], [13]). These early interpretations focused on $\Diamond$ as topological closure. Subsequent work demonstrated that $\Diamond$ can also be interpreted as the derivative[4] ([7], [3], [10], [1]). More recently, it has been shown that other topological operators—including border and boundary operators—can provide fruitful interpretations of various, usually non-normal, modal operators ([11]).

This paper attempts to continue this more recent line of inquiry by proposing a modal logic for isolated points. When $\mathcal{X} = \langle X, \tau \rangle$ is a topological space and $S \subseteq X$, x is an *isolated point* of S if there is an open neighborhood U of x such that $U \cap S = \{x\}$. We demonstrate that the $[i]$ operator introduced in [6] (where it is intended to model a notion of factive ignorance) can be spatially interpreted as an isolated points operator.

In §2, the basic syntax, axiomatic system and relational semantics are introduced briefly. Due to the relative lack of algebraic structure of the isolated points operator, instead of immediately focussing on topologies, we begin, in §3, by considering neighborhood systems.[5] As neighborhood systems are a generalization

[4] The *derivative* of a set S, $d(S)$, is the set of *limit points* of S.

[5] Some authors use the term *neighborhood system* to refer only to those families of neighborhoods that give rise to a topological space. Our usage will be more liberal, allowing any set equipped with a neighborhood function to qualify.

of topologies, we similarly generalize the concept of an isolated point, showing that this notion can be logically captured by $[i]$. Finally, in §4, we transition to considering topological spaces, discrete spaces in particular, before concluding in §5.

2 Syntax, Proof System, and Relational Semantics

Take $Prop$ to be a countably infinite set of propositional variables. The set $Form$ of well-formed formulas of the language $\mathcal{L}^i$ is recursively defined:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid [i]\alpha$$

for $p \in Prop$.

2.1 Axiom System

The basic axiomatic proof system, $\mathbf{S^i}$, as defined in [4] (where it is referred to as $\mathbf{L^i}$), is as follows:

Definition 1 (Proof System $\mathbf{S^i}$).
(Taut) All instances of propositional tautologies
(A1) $[i]\varphi \to \varphi$
(A2) $([i]\varphi \wedge [i]\psi) \to [i](\varphi \vee \psi)$
(MP) From $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *infer* $\vdash \psi$
(R1) From $\vdash \varphi \to \psi$ *infer* $\vdash \varphi \to ([i]\psi \to [i]\varphi)$

Our notions of *derivation*, *theorem*, and *consistency* are the usual ones.

Proposition 1. *The following are all theorems of* $\mathbf{S^i}$*:*

a. $([i]\varphi \wedge [i]\psi) \to [i](\varphi \wedge \psi)$
b. $[i](\varphi \vee \psi) \to ([i]\varphi \vee [i]\psi)$
c. $[i]\varphi \to [i][i]\varphi$

In addition, the rule allowing $\vdash [i]\varphi \leftrightarrow [i]\psi$ *from* $\vdash \varphi \leftrightarrow \psi$ *is derivable.*

2.2 Relational Semantics

Using relational semantics, satisfaction for $[i]$-formulas is defined as follows:

$$M, w \models [i]\varphi \text{ iff } (M, w \models \varphi \text{ and } \forall w' \neq w(wRw' \text{ implies } M, w' \not\models \varphi))$$

Theorem 1 ([4]). $\mathbf{S^i}$ *is sound and strongly complete with respect to the class of all relational frames.*

3 Neighborhood semantics for $\mathbf{S^i}$

As mentioned above, the ultimate goal of this paper is to make strides toward a topological interpretation of $[i]$. In particular, we suggest interpreting $[i]$ as an isolated points operator. Recall the definition of an isolated point in a topological space:

Definition 2 (Isolated Point). *Let $\mathcal{X} = \langle X, \tau\rangle$ be a topological space and $S \subseteq X$. x is an* isolated point *of S if there is an open neighborhood U of x such that $U \cap S = \{x\}$.*

Instead of beginning immediately with a purely topological semantics, we start by providing a more general semantic account in terms of neighborhood systems (we follow [9] and [2], for instance, in the treatment of neighborhood semantics).

Definition 3 (Neighborhood Frame). *A* neighborhood frame *is a pair $\langle X, N\rangle$ such that $X \neq \emptyset$ and $N : X \to \wp(\wp(X))$. A* neighborhood model *is a pair $\langle F, V\rangle$, where F is a neighborhood frame and $V : Prop \to \wp(X)$ is a valuation function.*

Generalizing the above definition of an isolated point to the context of neighborhood systems, where less mathematical structure is insisted upon, one can say that x is an isolated point of S if there is a neighborhood U of x, i.e., $U \in N(x)$, such that $U \cap S = \{x\}$. We can formalize this intuition in the definition of satisfaction for formulas in $\mathcal{L}^i$ with respect to neighborhood models. Given a model M and a formula α, the *truth set* of α in M, denoted $[\![\alpha]\!]^M$ is defined via recursion:

$$\begin{array}{ll}
[\![p]\!]^M & := V(p) \\
[\![\neg\varphi]\!]^M & := X \setminus [\![\varphi]\!]^M = ([\![\varphi]\!]^M)^c \\
[\![\varphi \wedge \psi]\!]^M & := [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\
[\![[i]\varphi]\!]^M & := \{x : \exists U \in N(x) \text{ s.t. } U \cap [\![\varphi]\!]^M = \{x\}\}
\end{array}$$

When no ambiguity can arise, the M superscript will be omitted. A formula is *valid* in a class of frames when it is true at all points in all models based on frames in the class. A set of formulas is *satisfiable* in a class of frames when there is a state in a model based on a frame in the class at which all the elements are true.

3.1 Semantic Insensitivities

In the context of relational frames and the semantics given in §2, $\mathcal{L}^i$ is *reflexive-insensitive*. That is, the satisfaction of $\mathcal{L}^i$-formulas in a model $M = \langle W, R, V\rangle$ is not affected when arbitrary elements from id_W are either added to, or removed from, R. In the neighborhood context, there are similar insensitivities.

In particular, because the definition of $[\![\cdot]\!]$ utilizes only sets of each $N(x)$ that contain x, the addition or removal of sets that do not contain x will be immaterial.

For a given neighborhood frame $\langle X, N\rangle$, consider the set

$$S_x := \{Y \in \wp(X) : x \notin Y\}$$

for each $x \in X$. (Here, we are following the notation of [5], which was concerned with neighborhood semantics for a different logic that was also insensitive to reflexivity in the relational setting.)

Then, given a neighborhood frame $F = \langle X, N\rangle$, construct the frames $F = \langle X, N^+\rangle$ and $F = \langle X, N^-\rangle$, where N^+ and N^- are defined as follows for all $x \in X$:

$$N^+(x) := N(x) \cup S_x$$

$$N^-(x) := N(x) \setminus S_x$$

When M is a neighborhood model, M^+ (M^-) is that model identical to M, but with N^+ (N^-) replacing N.

Proposition 2. *Let M be a neighborhood model. Then M, M^+, and M^- (as well as the intermediate models) are all pointwise equivalent. That is,*

$$[\![\alpha]\!]^{M^-} = [\![\alpha]\!]^{M} = [\![\alpha]\!]^{M^+}$$

for all $\alpha \in Form$.

However, in the current setting there are additional sensitivities that one can utilize.

Definition 4 (Supplemented Neighborhood System). *A neighborhood frame is* supplemented *when its neighborhood function is closed under supersets: for every x, if $Y \subseteq N(x)$ and $Y \subseteq Z$, then $Z \subseteq N(x)$.*

Given a neighborhood frame, $F = \langle X, N\rangle$, let $F^s = \langle X, N^s\rangle$ be the supplementation *of F when, for all $x \in X$:*

$$N^s(x) = \{Y \subseteq \wp(X) : \exists U \subseteq Y \text{ s.t. } U \in N(x)\}$$

For a model $M = \langle F, V\rangle$, let $M^s = \langle F^s, V\rangle$.

Remark 1. Let M be a model and M^s its supplementation. Then it is not necessarily the case that

$$[\![\alpha]\!]^{M} = [\![\alpha]\!]^{M^s}$$

The countermodels demonstrating this observation make use of supplementing some $N(x)$ containing at least one set from S_x and thereby adding sets to $N(x)$ not in S_x. (For instance, consider some state x such that $N(x) = \{\emptyset\}$. Then for no φ will it be the case that $x \in [\![\varphi]\!]^M$. However, since $\{x\} \in N^s(x)$, $x \in [\![\varphi]\!]^{M^s}$ for every φ such that $x \in [\![\varphi]\!]^{M^s}$. This is discussed further in §4, below.)

However, if no such sets are present in any $N(x)$ (for instance, as in neighborhood filters in topological spaces), then supplementation will not affect satisfaction.

Definition 5 (Anchored Neighborhood System). *A neighborhood function (and, hence, the resulting system) is* anchored *when, for every point $x \in X$,*

$$\forall U \in N(x)(x \in U)$$

(Note that we do not force $N(x) \neq \emptyset$ in order to be anchored.)

Proposition 3. *Let M be an anchored neighborhood model. Then*

$$[\![\alpha]\!]^M = [\![\alpha]\!]^{M^s}$$

Proof. Induction on α. We omit all but the modal case.

If $x \in [\![[i]\varphi]\!]^M$, then there is some $U \in N(x)$ s.t. $U \cap [\![\varphi]\!]^M = \{x\}$. Since $U \in N^s(x)$ and $[\![\varphi]\!]^M = [\![\varphi]\!]^{M^s}$, by the induction hypothesis, $x \in [\![[i]\varphi]\!]^{M^s}$.

In the other direction, if $x \in [\![[i]\varphi]\!]^{M^s}$, there is a $U \in N^s(x)$ s.t. $U \cap [\![\varphi]\!]^{M^s} = \{x\}$. Thus, there must have been some $U_1 \in N(x)$ s.t. $U_1 \subseteq U$. But, since N is anchored, $x \in U_1$, and so $U_1 \cap [\![\varphi]\!]^{M^s} = \{x\}$. By the induction hypothesis, $[\![\varphi]\!]^{M^s} = [\![\varphi]\!]^M$, so $U_1 \cap [\![\varphi]\!]^M = \{x\}$. Hence, $x \in [\![[i]\varphi]\!]^M$.

In particular, Proposition 3 guarantees that

$$[\![\alpha]\!]^{M^{-s}} = [\![\alpha]\!]^M$$

So every model will be pointwise equivalent to some supplemented model.

3.2 Soundness and Completeness

Using standard methods, characterization results for $\mathbf{S^i}$, with respect to the given neighborhood semantics, are readily obtained. A logic is said to be *sound* with respect to a class of frames when all theorems of the logic are valid in the class. A logic is *complete* with respect to a class of frames when every consistent formula is satisfiable in the class. A logic is *strongly complete* with respect to a class when every consistent set of formulas is satisfiable in the class.

A neighborhood frame $\langle X, N \rangle$ is said to be *closed under intersections* when, for every $x \in X$, if $U \in N(x)$ and $V \in N(x)$, then $U \cap V \in N(x)$.

Theorem 2 (Soundness). $\mathbf{S^i}$ *is sound with respect to the class of neighborhood frames that are closed under intersections.*

Proof. The proof is standard, and proceeds by showing that all axioms are valid and rules preserve validity. We include only the cases unique to $\mathbf{S^i}$.

$[i]\varphi \to \varphi$: Assume $x \in [\![[i]\varphi]\!]$. Then $\exists U \in N(x)$ s.t. $U \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![\varphi]\!]$.

$([i]\varphi \wedge [i]\psi) \rightarrow [i](\varphi \vee \psi)$: Assume $x \in [\![[i]\varphi]\!]$ and $x \in [\![[i]\psi]\!]$. Then $\exists U_1 \in N(x)$ s.t. $U_1 \cap [\![\varphi]\!] = \{x\}$ and $\exists U_2 \in N(x)$ s.t. $U_2 \cap [\![\psi]\!] = \{x\}$. Since N is closed under intersections, $U_1 \cap U_2 \in N(x)$. But $(U_1 \cap U_2) \cap [\![\varphi \vee \psi]\!] = \{x\}$, so $x \in [\![[i](\varphi \vee \psi)]\!]$.

From $\vdash \varphi \rightarrow \psi$ infer $\vdash \varphi \rightarrow ([i]\psi \rightarrow [i]\varphi)$: Assume the validity of $\varphi \rightarrow \psi$. Then $[\![\varphi]\!] \subseteq [\![\psi]\!]$ in all models. Assume, further, $x \in [\![\varphi]\!]$. If $x \in [\![[i]\psi]\!]$, then $U \cap [\![\psi]\!] = \{x\}$, for some $U \in N(x)$. Since $[\![\varphi]\!] \subseteq [\![\psi]\!]$, $U \setminus \{x\} \subseteq X \setminus [\![\psi]\!] \subseteq X \setminus [\![\varphi]\!]$. Hence, $U \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![[i]\varphi]\!]$.

Theorem 3 (Completeness). $\mathbf{S^i}$ *is strongly complete with respect to the class of neighborhood frames that are anchored and closed under intersections.*

Proof. We give a canonical model construction. (The argument for completeness given the canonical model is standard.) Given the set of maximal $\mathbf{S^i}$-consistent sets, $\Sigma_{\mathbf{S^i}}$, define

$$|\alpha| = \{x \in \Sigma_{\mathbf{S^i}} : \alpha \in x\}$$

Construct the canonical model $M^{\mathbf{S^i}} = \langle X^{\mathbf{S^i}}, N^{\mathbf{S^i}}, V^{\mathbf{S^i}} \rangle$ as follows:

- $X^{\mathbf{S^i}} := \Sigma_{\mathbf{S^i}}$
- for each $x \in X^{\mathbf{S^i}}$, $N(x) := \{|\neg\varphi| \cup \{x\} : [i]\varphi \in x\}$
- for each $p \in \mathit{Prop}$, $V^{\mathbf{S^i}}(p) = |p|$

(The derivable rule mentioned in Proposition 1 ensures that N is well-defined.) A straightforward induction then demonstrates that, for all $\alpha \in \mathit{Form}$,

$$[\![\alpha]\!]^{\mathbf{S^i}} = |\alpha|$$

The only non-trivial case is that of the modality. (In what follows, we omit all $\mathbf{S^i}$ superscipts.)

If $x \in |[i]\varphi|$, then, by definition, $|\neg\varphi| \cup \{x\} = (X \setminus |\varphi|) \cup \{x\} \in N(x)$. By the induction hypothesis, $|\varphi| = [\![\varphi]\!]$, so $(X \setminus [\![\varphi]\!]) \cup \{x\} \in N(x)$. Since $x \in |[i]\varphi \rightarrow \varphi|$, $x \in |\varphi| = [\![\varphi]\!]$. Finally, $((X \setminus [\![\varphi]\!]) \cup \{x\}) \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![[i]\varphi]\!]$.

If $x \in [\![[i]\varphi]\!]$, then there is some $U \in N(x)$ such that $U \cap [\![\varphi]\!] = \{x\}$ (hence, $x \in [\![\varphi]\!] = |\varphi|$, by the induction hypothesis). By construction, $U = |\neg\psi| \cup \{x\}$ for some ψ such that $x \in |[i]\psi|$ (and $x \in |\psi|$). But then $|\neg\psi| \subseteq |\neg\varphi|$, and so $|\varphi| \subseteq |\psi|$, meaning that $\vdash \varphi \rightarrow \psi$. Therefore, $\vdash \varphi \rightarrow ([i]\psi \rightarrow [i]\varphi)$. Since $x \in |\varphi|$, $x \in |[i]\psi \rightarrow [i]\varphi|$. And, because $x \in |[i]\psi|$, $x \in |[i]\varphi|$.

Lastly, the model is both closed under intersections and anchored. Anchoring is by construction.

For closure under intersections, assume that $U_1, U_2 \in N(x)$. Then, $U_1 = |\neg\varphi_1| \cup \{x\}$ and $U_2 = |\neg\varphi_2| \cup \{x\}$ with $x \in |[i]\varphi_1|$ and $x \in |[i]\varphi_2|$. Since x is a maximal $\mathbf{S^i}$-consistent set, $x \in |[i]\varphi_1 \wedge [i]\varphi_2|$, and so $x \in |[i](\varphi_1 \vee \varphi_2)|$. By construction, $|\neg(\varphi_1 \vee \varphi_2)| \cup \{x\} \in N(x)$. But $|\neg(\varphi_1 \vee \varphi_2)| = |\neg\varphi_1| \cap |\neg\varphi_2|$, and $U_1 \cap U_2 = (|\neg\varphi_1| \cap |\neg\varphi_2|) \cup \{x\}$, so $U_1 \cap U_2 \in N(x)$.

Corollary 1. $\mathbf{S^i}$ *is strongly complete with respect to the class of neighborhood frames that are anchored, closed under intersections, and supplemented.*

Proof. Since $M^{\mathbf{S^i}}$ is anchored, it is pointwise equivalent to its supplementation, from Proposition 3.

In addition, making use of the standard conversion between relational frames and augmented neighborhood structures, a completeness theorem can also be obtained with respect to the class of all augmented neighborhood frames.

Definition 6 (Augmented Neighborhood System). *A neighborhood function (and, hence, the resulting system) is* augmented *when, for every point* $x \in X$, $N(x)$ *is supplemented and* $\bigcap N(x) \in N(x)$.

Lemma 1. *For every relational model, there is a pointwise equivalent neighborhood model that is augmented.*

Proof. Let $M = \langle W, R, V \rangle$ be an arbitrary relational model. Define the function $N_R : W \to \wp(\wp(W))$ as

$$N_R(w) := \{X : R(w) \subseteq X\}$$

where $R(w) = \{y \in W : wRy\}$. Let $M^N = \langle W, N_R, V \rangle$. Note that N_R is augmented.

For all wffs α,

$$M, w \models \alpha \text{ iff } w \in [\![\alpha]\!]^{M^N}$$

This is, again, an induction on α and only the modal case will be discussed.

Assume $M, w \models [i]\varphi$. Then $M, w \models \varphi$ and $\forall z \neq w$, wRz implies $M, z \not\models \varphi$. Hence, $w \in [\![\varphi]\!]$ (from the induction hypothesis) and $R(w) \setminus \{w\} \subseteq W \setminus [\![\varphi]\!]$. Since N_R is augmented, $R(w) \cup \{w\} \in N_R$, and $(R(w) \cup \{w\}) \cap [\![\varphi]\!] = \{w\}$, so $w \in [\![[i]\varphi]\!]$.

Assume now that $M, w \not\models [i]\varphi$. Then either $M, w \not\models \varphi$ or, for some $z \neq w$ s.t. wRz, $M, z \models \varphi$.

If $M, w \not\models \varphi$ then, by the induction hypothesis, $w \notin [\![\varphi]\!]$, and so $w \notin [\![[i]\varphi]\!]$.

Otherwise, assume that $M, w \models \varphi$ and $M, z \models \varphi$ for some $z \neq w$ s.t. wRz. Then $\{z\} \subseteq R(w)$ and $\{w, z\} \subseteq [\![\varphi]\!]$. Therefore, $\{z\} \subseteq U \cap [\![\varphi]\!]$ for all $U \in N_R(w)$, so $w \notin [\![[i]\varphi]\!]$.

Clearly, if the original model was reflexive, then the resulting augmented model is anchored.

Lemma 2. *For every augmented neighborhood model, there exists a pointwise equivalent relational model.*

Proof. Let $M = \langle X, N, V \rangle$ be an arbitrary augmented neighborhood model. Define the relational model $M^R = \langle X, R_N, V \rangle$ such that xR_Ny iff $y \in \bigcap N(x)$. Then, for all wffs α,

$$x \in [\![\alpha]\!]^M \text{ iff } M^R, x \models \alpha$$

Induction on α.

Assume $x \in [\![[i]\varphi]\!]$. Then there is a $U \in N(x)$ such that $U \cap [\![\varphi]\!] = \{x\}$. Hence, from the induction hypothesis, $M^R, x \models \varphi$. Moreover, $\bigcap N(x) \subseteq U$, so $\bigcap N(x) \setminus \{x\} \subseteq X \setminus [\![\varphi]\!]$. Therefore, for any $y \neq x$ such that $y \in \bigcap N(x)$, $y \in X \setminus [\![\varphi]\!]$, so $M^R, y \not\models \varphi$, by the induction hypothesis. Hence, $M^R, x \models [i]\varphi$.

In the other direction, if $M^R, x \models [i]\varphi$, then $M^R, x \models \varphi$ and $\forall y \neq x$, xR_Ny implies $M, y \not\models \varphi$. By the induction hypothesis, $x \in [\![\varphi]\!]$ and $\forall y \neq x$, if xR_Ny, then $y \notin [\![\varphi]\!]$. But xR_Ny iff $y \in \bigcap N(x)$. Hence, $\bigcap N(x) \setminus \{x\} \subseteq X \setminus [\![\varphi]\!]$. Let $U = \bigcap N(x) \cup \{x\}$. Then $U \in N(x)$, since $N(x)$ is supplemented. Moreover, $U \cap [\![\varphi]\!] = \{x\}$, so $x \in [\![[i]\varphi]\!]$.

If the original neighborhood system was anchored, then the resulting relational model is reflexive.

Corollary 2. $\mathbf{S^i}$ *is sound and (strongly) complete with respect to the class of all augmented neighborhood frames and all anchored, augmented neighborhood frames.*

Proof. For the anchored, augmented neighborhood frames, strong completeness follows from taking the reflexive closure of the canonical model used in the proof of Theorem 1 (as defined in [4]) along with Lemma 1.

4 Discrete Neighborhood Systems

Definition 7. *A neighborhood system is* discrete *when* $\{x\} \in N(x)$*, for every* $x \in X$*.*

Consider the following axiom schema:

$$\varphi \leftrightarrow [i]\varphi \tag{Disc}$$

Call **SDisc** the system obtained by adding (Disc) to $\mathbf{S}^i$.

In the presence of (Disc), no other modal axioms are necessary and neither is the rule (R1).

Proposition 4. SDisc *can be axiomatized by the following:*
(Taut) All instances of propositional tautologies
(Disc) $\varphi \leftrightarrow [i]\varphi$
(MP) From $\vdash \varphi$ *and* $\vdash \varphi \to \psi$ *infer* $\vdash \psi$

Proposition 5. SDisc *is valid on a relational frame* F *if and only if, for each* w*, if* wRz*, then* $w = z$*.*

Corollary 3. **SDisc** *is sound with respect to the class of relational frames in which, for each w, if wRz, then $w = z$.*

Theorem 4. **SDisc** *is strongly complete with respect to the class of relational frames in which, for each w, if wRz, then $w = z$.*

Proof. This is easily seen by inspecting the canonical model for $\mathbf{S^i}$—as given in [4]—and observing that $[i]\top$ is an element of each maximal consistent set and will, therefore, have an empty accessibility relation.

In terms of neighborhood systems, a characterization result for **SDisc** is also straightforward.

Theorem 5. **SDisc** *is sound and strongly complete with respect to the class of discrete neighborhood systems. (Hence, due to the semantic insensitivities noted above, also with respect to anchored, discrete, supplemented neighborhood systems.)*

Proof. Soundness is immediate.

For completeness, one need only look at the canonical model construction in the proof of Theorem 3 and observe that, in the presence of (Disc), $[i]\top \in x$, for every $x \in X^{\mathbf{SDisc}}$. Hence, $|\bot| \cup \{x\} = \{x\} \in N(x)$.

4.1 Discrete Topologies

We can conclude the main section of the paper by (finally) transitioning to topologies proper.

Recall that a topo-model is a pair $M = \langle \mathcal{X}, V \rangle$ where $\mathcal{X} = \langle X, \tau \rangle$ is a topological space and $V : Prop \to \wp(X)$ is a valuation function. A formula φ is true in M when it is true at every $x \in \mathcal{X}$. φ is valid in $\mathcal{X}$ when it is true in every M based on $\mathcal{X}$. φ is valid in a class of topological spaces when φ is valid in every member of the class.

Satisfaction at points in a topo-model is defined as usual, with the clause for $[i]$ resembling closely the one given for neighborhood systems, but with reference to the topology τ rather than the neighborhood function N:

$$[\![[i]\varphi]\!] := \{x : \exists U \in \tau \text{ s.t. } U \cap [\![\varphi]\!] = \{x\}\}$$

With the semantics so defined, $\mathbf{S^i}$ is sound with respect to the class of all topo-models (the proof is fundamentally the same as that of Theorem 2). Moreover, **SDisc** is sound with respect to the class of all discrete topological spaces, since all singletons are open.

The results above, concerning discrete neighborhood spaces, can be transferred over to the topological setting to render a completeness result for **SDisc** as well.

Definition 8 (Neighborhood Topology). *A neighborhood function N is a* neighborhood topology *when the following conditions are met:*

a. If $S \in N(x)$, then $x \in S$;
b. each $N(x)$ is closed under supersets;
c. each $N(x)$ is closed under intersections;
d. for each $S \in N(x)$, there is a $T \subseteq S$ such that $T \in N(x)$ and, for each $y \in T$, $S \in N(y)$.

Moreover, given a neighborhood topology over a set X, the pair $\langle X, \tau\rangle$ is a topological space when[6]

$$U \in \tau \textit{ iff } \forall x \in U, U \in N(x)$$

Theorem 6. **SDisc** *is complete with respect to the class of all discrete topological spaces.*

Proof. Consider $M^{\mathbf{SDisc}}$, the canonical neighborhood model for **SDisc** (referred to in Theorem 5). The model is anchored and discrete. Let M^S be the supplementation of $M^{\mathbf{SDisc}}$. The frame of M^S is then a neighborhood topology. Consider the resulting topological space $\langle X, \tau\rangle$. Since, for each x, $\{x\} \in N(x)$, the topology is discrete. Let $M_{\mathcal{X}}$ be the topo-model obtained by adding V, the valuation function from M^S, to $\mathcal{X}$. A straighforward induction proves that M^S and $M_{\mathcal{X}}$ are pointwise equivalent.

5 Conclusion and Future Work

Thus far we have tried to argue that there is a plausible interpretation of $[i]$ as an isolated points operator in a variety of neighborhood systems, including those that correspond to discrete topologies. Immediately, there is the question of whether or not there exist intermediate logics (between $\mathbf{S^i}$ and **SDisc**) that characterize interesting classes of either neighborhood systems or topologies. It is not immediately obvious what such logics look like, or if any even exist.

It might be slightly more promising to examine extensions of $\mathbf{S^i}$ with only neighborhood systems in mind. For example, one can consider adding to $\mathbf{S^i}$ the axiom $\neg[i]\top$. Relationally, this has the effect of forcing all worlds to be non-reflexively serial (that is, for each x, there is a $y \neq x$ such that xRy). It is easy to see that, in neighborhood systems, this axiom forces a lack of discreteness (hence, the resulting logic is not an intermediate logic, but inconsistent with **SDisc**), and that it is sound and strongly complete with respect to the class of all neighborhood systems in which $\{x\} \notin N(x)$.[7]

We leave these questions for future work.

[6] See, for instance, [8].

[7] In the canonical model, the only way $\{x\}$ could be added to $N(x)$ is if there is a formula φ such that $[\![\neg\varphi]\!] \cup \{x\} = \{x\}$. This can only occur if either $[\![\neg\varphi]\!] = \emptyset$, which is ruled out by the new axiom, or if $[\![\neg\varphi]\!] = \{x\}$, but this is impossible because no formulas uniquely identify a maximal consistent set.

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References

1. G. Bezhanishvili, L. Esakia, and D. Gabelaia. Some results on modal axiomatization and definability for topological spaces. *Studia Logica*, 81(3):325–355, 2005.
2. B. Chellas. *Modal Logic: an Introduction.* Cambridge University Press, 1980.
3. L. Esakia. Intuitionistic logic and modality via topology. *Annals of Pure and Applied Logic*, 127(1):155–170, 2004.
4. D. Gilbert, E. Kubyshkina, M. Petrolo, and G. Venturi. Logics of Ignorance and Being Wrong. *Logic Journal of the IGPL*, 30(5):870–885, 07 2021.
5. D. Gilbert and G. Venturi. Neighborhood semantics for logics of unknown truths and false beliefs. *Australasian Journal of Logic*, 14(1):246–267, 2017.
6. E. Kubyshkina and M. Petrolo. A logic for factive ignorance. *Synthese*, 198:5917–5928, 2021.
7. J. C. C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45(1):141–191, 1944.
8. B. Mendelson. *Introduction to Topology.* Allyn and Bacon, Boston, 3d edition, 1974.
9. E. Pacuit. *Neighborhood Semantics for Modal Logic.* Springer, Cham, Switzerland, 2017.
10. V. Shehtman. Derived sets in euclidean spaces and modal logic. Technical report, The University of Amsterdam, 1990.
11. C. Steinsvold. A note on logics of ignorance and borders. *Notre Dame Journal of Formal Logic*, 49(4):385–392, 2008.
12. T.-C. Tang. Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication. *Bulletin (new series) of the American Mathematical Society*, 44(10):737–744, 1938.
13. A. Tarski. Der Aussagenkalkül und die Topologie. *Fundamenta Mathematicae*, 31:103–134, 1938.